Faculty of Power and Aeronautical Engineering
WARSAW UNIVERSITY OF TECHNOLOGY

System Identification in Aerospace Engineering

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Basic Information

- Piotr Lichota, PhD DSc
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  - consultations: Monday 13^{15}-14^{00}, r. 106 (IAAM, Nowowiejska 24)

Materials:

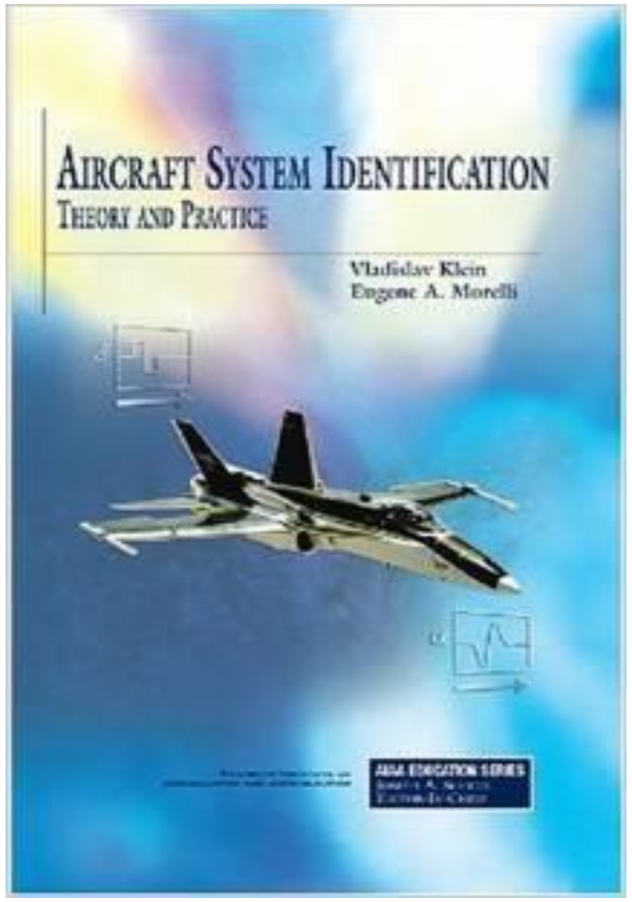
- **website:** https://www.meil.pw.edu.pl/zm/ZM/Dydaktyka/Prowadzone-przedmioty/System-Identification-In-Aerospace-Engineering
- **login:** SIAE
- **password:** SysId

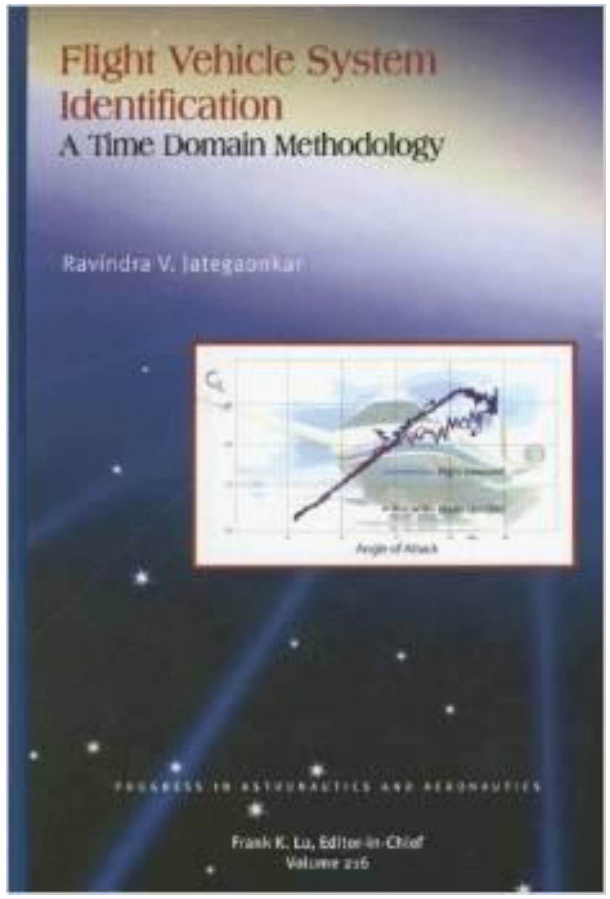
Rules:

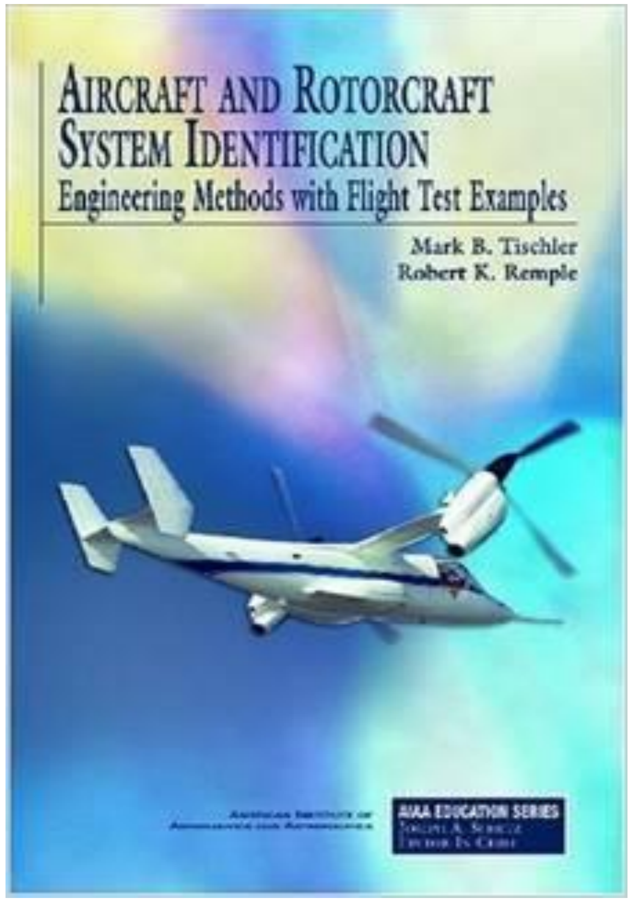
- Multiple choice test (20 question, 4 answers), 25pts
- Laboratory (4 exercises), 4 pts
- Attendance, 1 pt (6att)

Points (above)	Grade
12.50	3.0
15.00	3.5
17.50	4.0
20.00	4.5
22.50	5.0

Literature







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- Ljung L.: "System Identification: Theory for the User", Prentice Hall, Upper Saddle River, 1998
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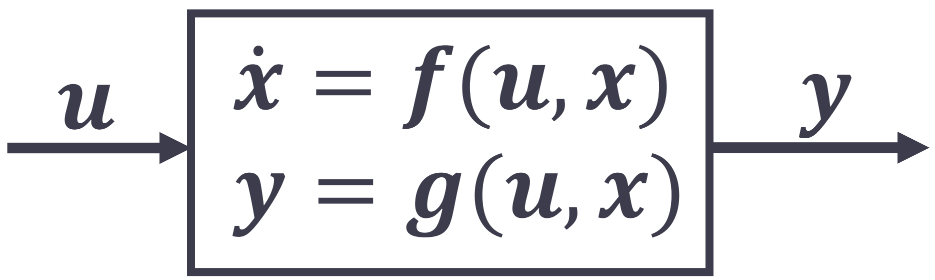
- Jategaonkar, R. V., „Flight Vehicle System Identification: A Time Domain Methodology," Progess in Astronautics and Aeronautics, AIAA, Reston, VA (USA), 2006.
- Tischler, M.B., Remple, R. K., „Aircraft and Rotorcraft System Identification: Engineering Methods with Flight Test Examples", AIAA Education Series, AIAA, Reston, VA (USA), 2006.

Topics

- Introduction
- Mathematical model
- Experiment planning
- Measurement and data compatibility check
- Equation error methods
- Output error methods
- Filter error methods
- Identification from frequency responses
- Artificial neural networks
- Online identification
- Dynamically unstable aircraft identification
- Validation

System identification



Task	Known	Unknown
Identification	u, y	f, g
Control	f, g, y	u
Simulation	f, g, u	y

„Determination, on the basis of observation of input and output, of a system within a specified class of systems to which the system under test is equivalent."

Lofti Zadeh

- Parameter estimation – known model structure, unknown parameter values
- System identification – unknown model structure, unknown parameter values

System identification

- Aims
  - Obtaining mathematical models that can be used for
  - Understanding cause-effect relationships that underlies a physical phenomenon
  - Investigating system performance characteristics
  - Verifying results obtained from theory/wind tunnel/CFD
  - Developing aerodynamic databases for flight simulators
  - Expanding flight envelope during prototype testing
  - Developing in-flight simulators
  - Designing control laws and flying qualities evaluation
  - Flight path reconstruction
  - Diagnostics, adaptive control and reconfiguration

- Assumptions:
  - True state of the system is deterministic
  - Physical principles that underlay the process can be modelled
  - It is possible to perform specific experiments
  - Measurements of the system inputs and object response are available

System Identification

- 1777 - Daniel Bernoulli
  - Introduces the concept of the Maximum Likelihood function: „The most probable choice between several discrepant observations and the formation therefrom of the most likely induction”

- 1795 – Carl Friedrich Gauss
  - Develops and applies the Least Squares Method for celestial bodies orbits estimation

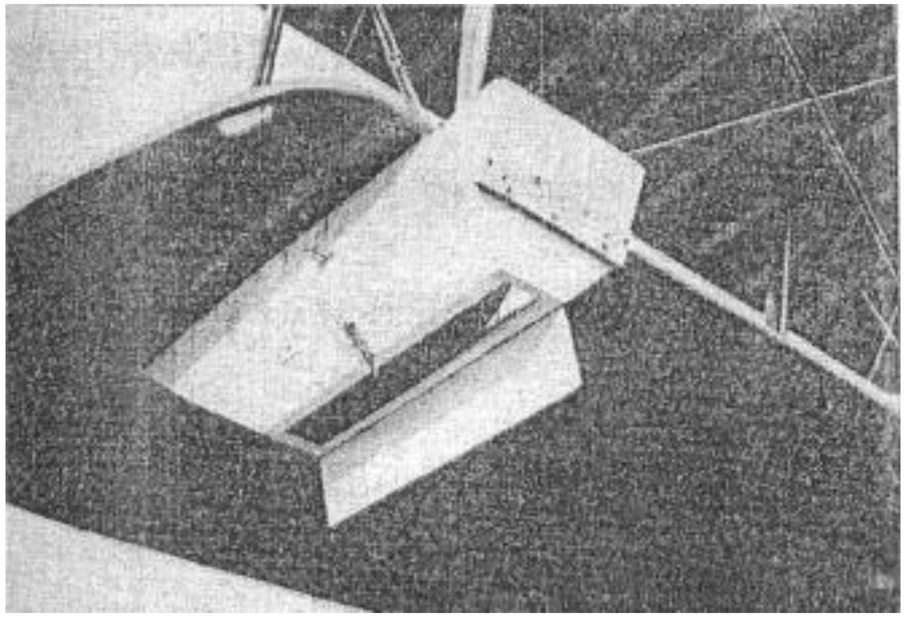
- 1912-1922 – Sir Ronald Aylmer Fisher
  - Develops and popularizes Maximum Likelihood Principle

Early days

- Norton 1919-1923

1. Sand boxes mounted under the wing (2x150lb) at 14.7ft from the longitudinal axis
2. Quick emptying of one box (less than 0.5s) - the plane starts to roll
3. After reaching a certain tilt angle (90deg), empty the second box

Curtiss JN4



- The maneuver was performed multiple times
- Registration of linear and angular velocities
- Evaluations of stability and control derivatives based on very simple formulas

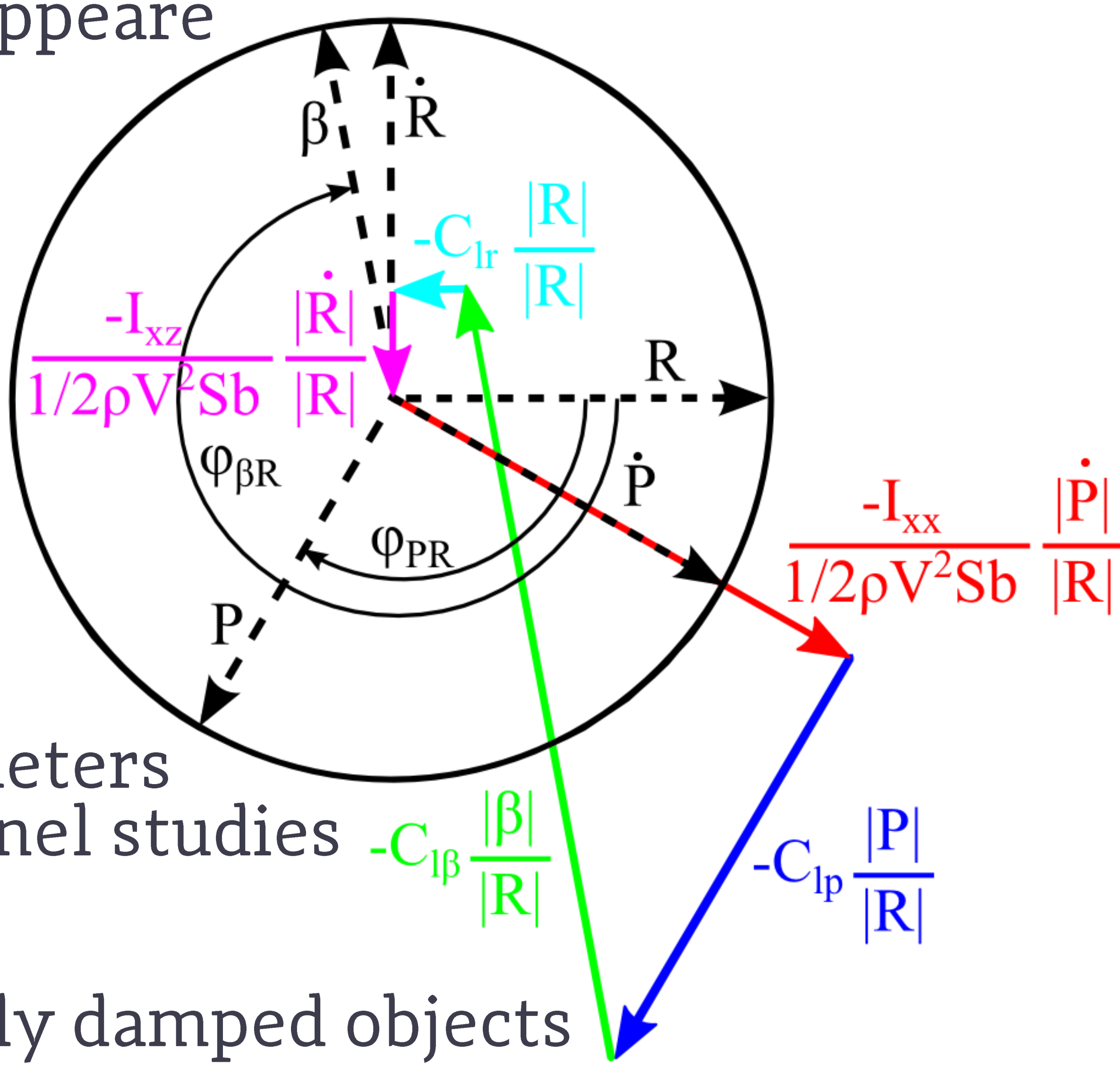
$$m = 72\text{slug} \approx 1050\text{kg}$$

$$L = 150\text{lb} \cdot 14.7\text{ft}$$

$$L_P P = L/m$$

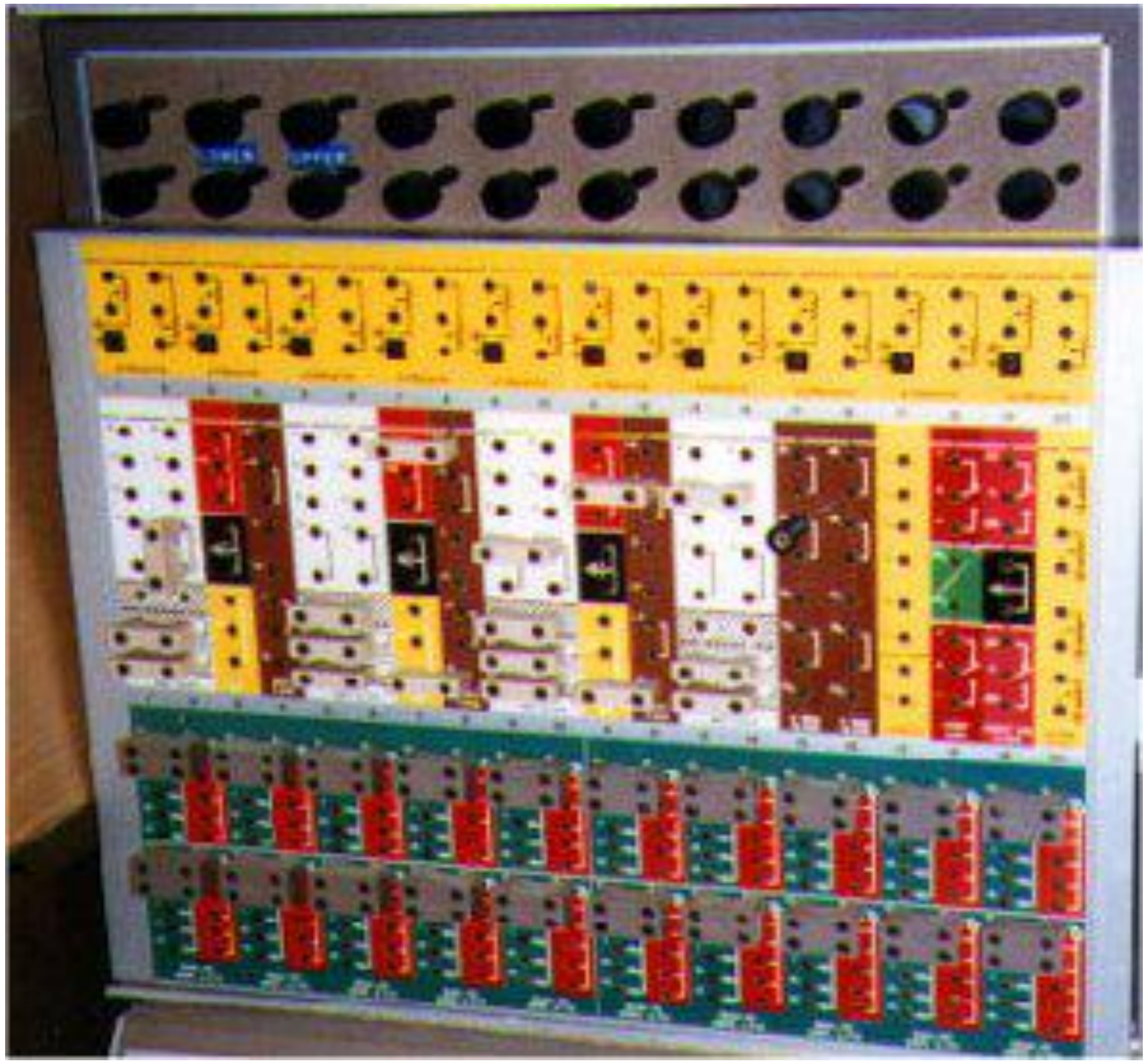
Time vector method (50s)

- Graphical method
- Exciting object vibrations by using a sharp rectangular signal
- Vibrations registration until they disappeare
- Magnitude and phase evaluation for individual degrees of freedom
- Drawing and adding vectors for individual degrees of freedom
- Calculation of selected model parameters
- The method allowed the estimation of only two (e.g. C_{lp}, $C_{l\beta}$) of three parameters - the third one is known, e.g. from tunnel studies
- Time-consuming process
- Difficulties with application to strongly damped objects



Analog matching (60s)

- Solving simplified equations of motion by using analog computer
- Manually select model parameter values to match model response and measurement data
  - Estimation accuracy depends on the operator's sensitivity in tuning parameters
  - Time-consuming process
  - The method allowed for the identification of only a few selected parameters (with the greatest impact on the model)
  - The results of the method depended very strongly on the quality of the recorded data

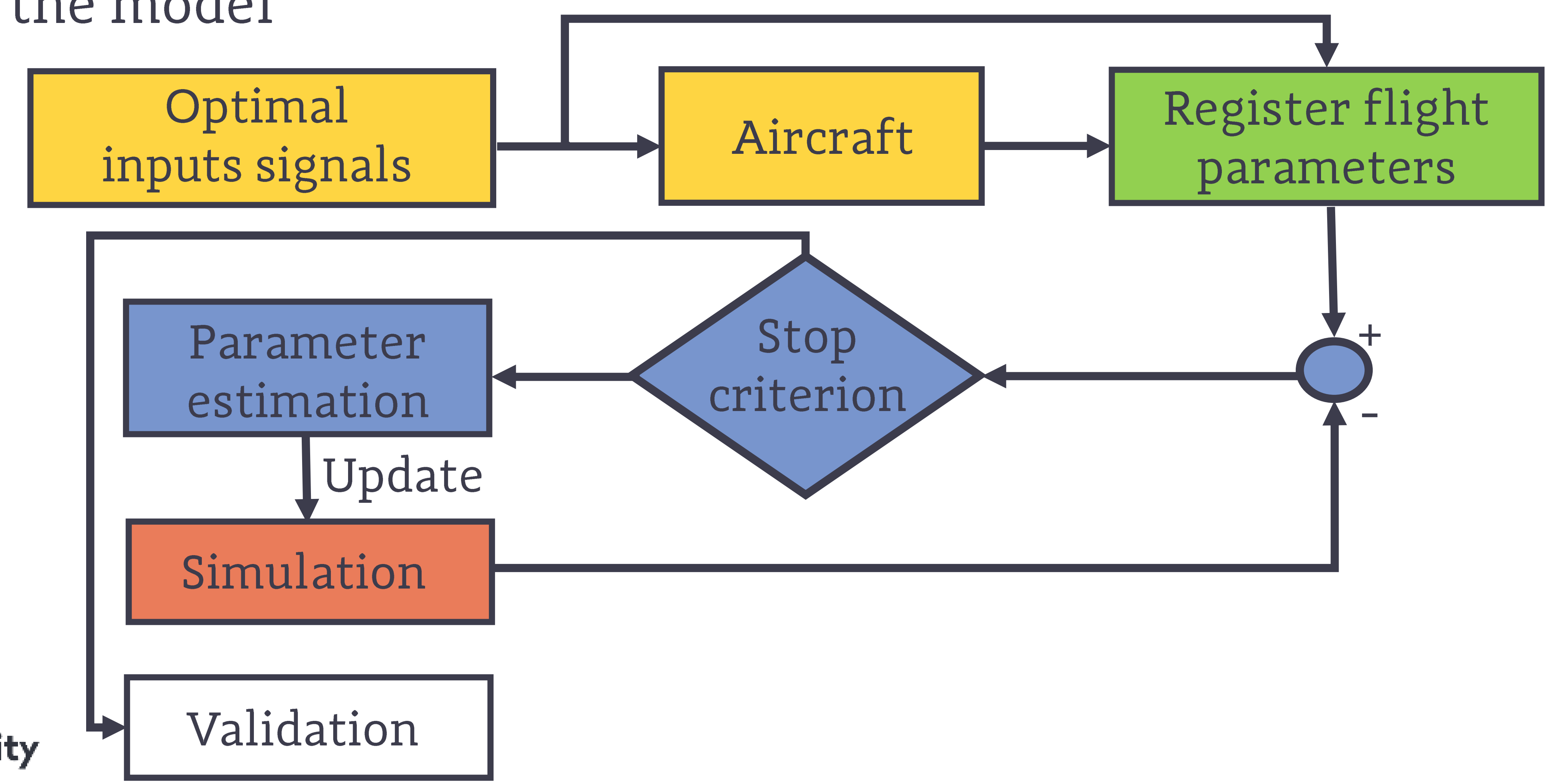


Modern system identification

- Billerud-IBM Project in 1963-1966
  - The use of computers to manage production at the Billerud paper mill (600 tons)
  - 1965 - Karl Åström and Torsten Bohlin implement the maximum likelihood principle on a digital computer for system identification

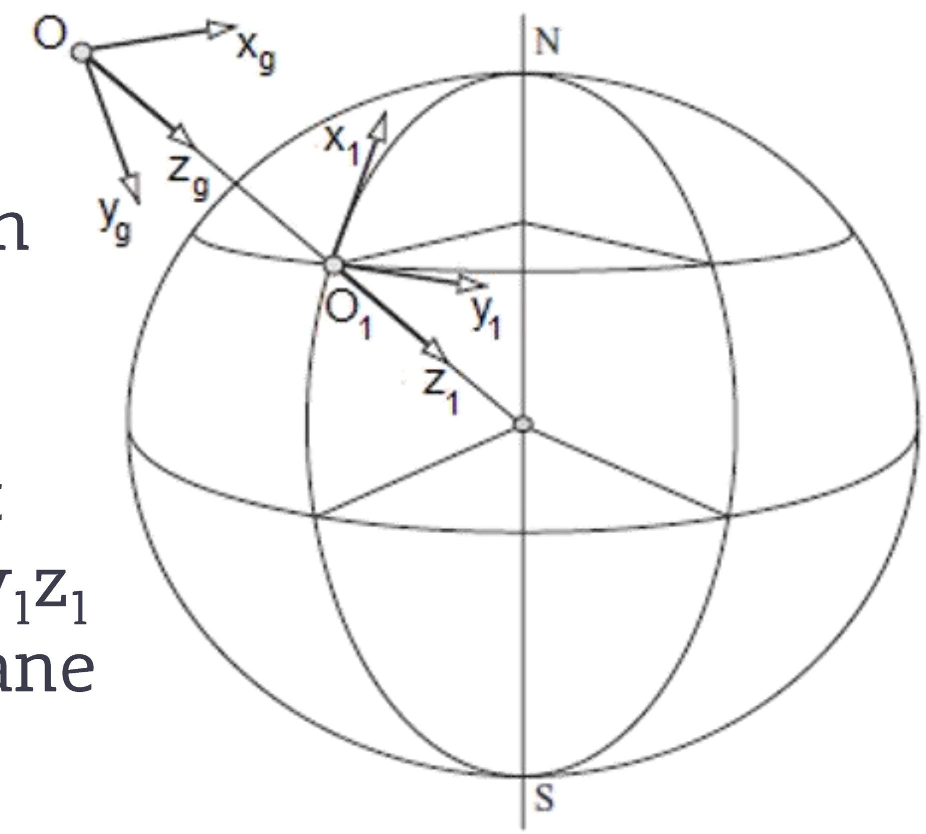
4-M methodology

- Manoeuvre – plan and perform the experiment
- Measurement – measure and register the signals (flight parameters, control surfaces deflections)
- Method – chose appropriate identification method, estimate
- Model – build a mathematical model, evaluate model response
- Validate the model



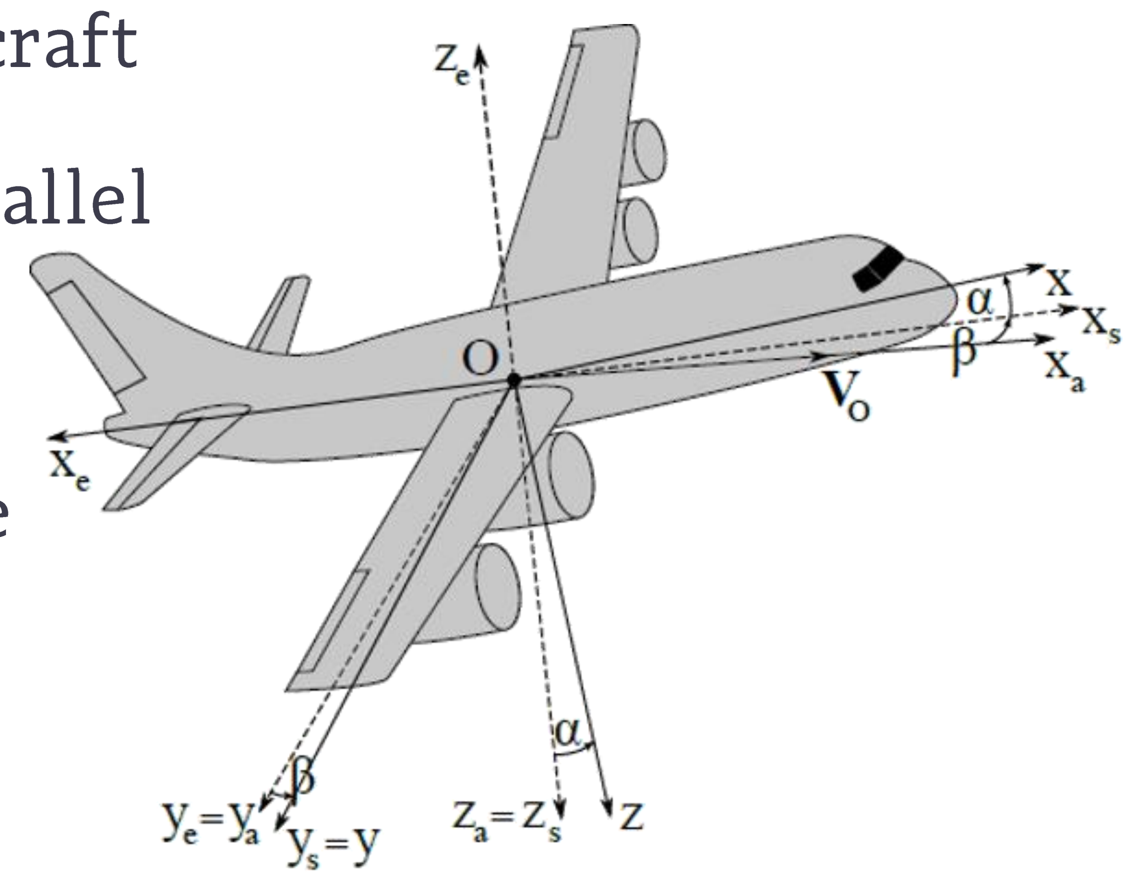
Coordinate systems

- Earth fixed reference frame $O_1x_1y_1z_1$
  - Point O_1 located on the surface of Earth
- $O_1x_1y_1$ plane is tangent to the surface of Earth



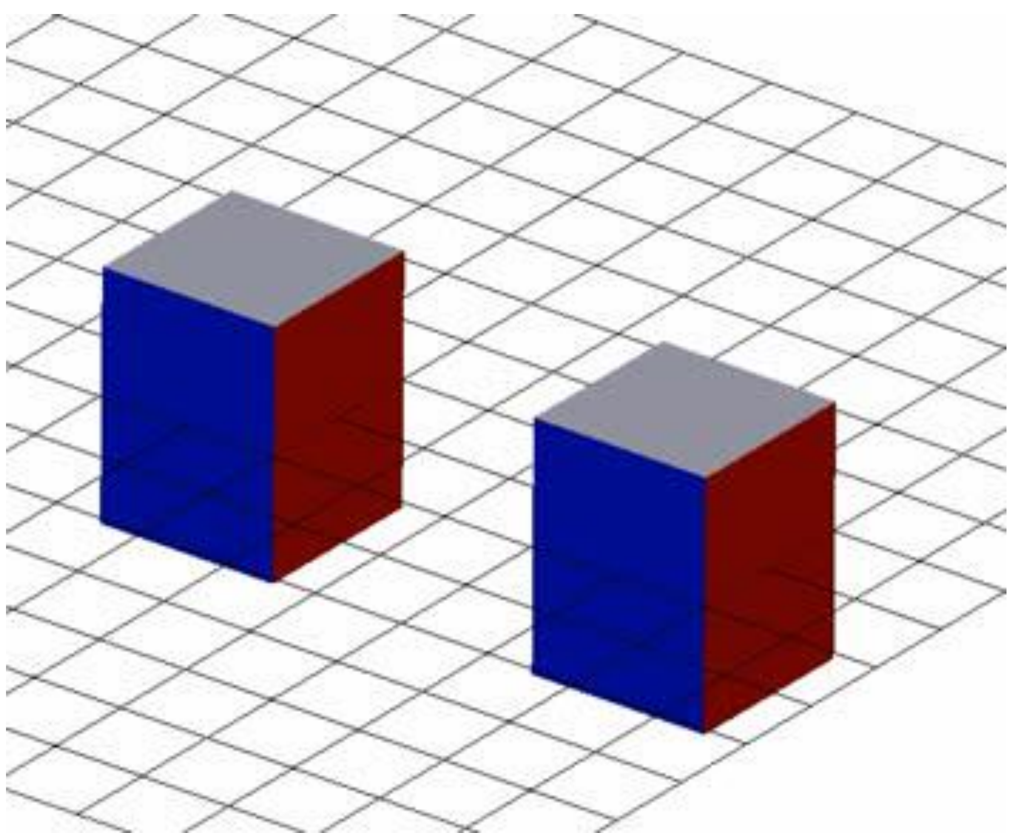
- Vehicle-carried Earth axes $Ox_gy_gz_g$
  - Point O is an arbitrary point of the aircraft
  - When t=0, $Ox_gy_gz_g$ axes coincide with $Ox_1y_1z_1$
  - The $Ox_gy_gz_g$ axes are parallel to $Ox_1y_1z_1$ plane

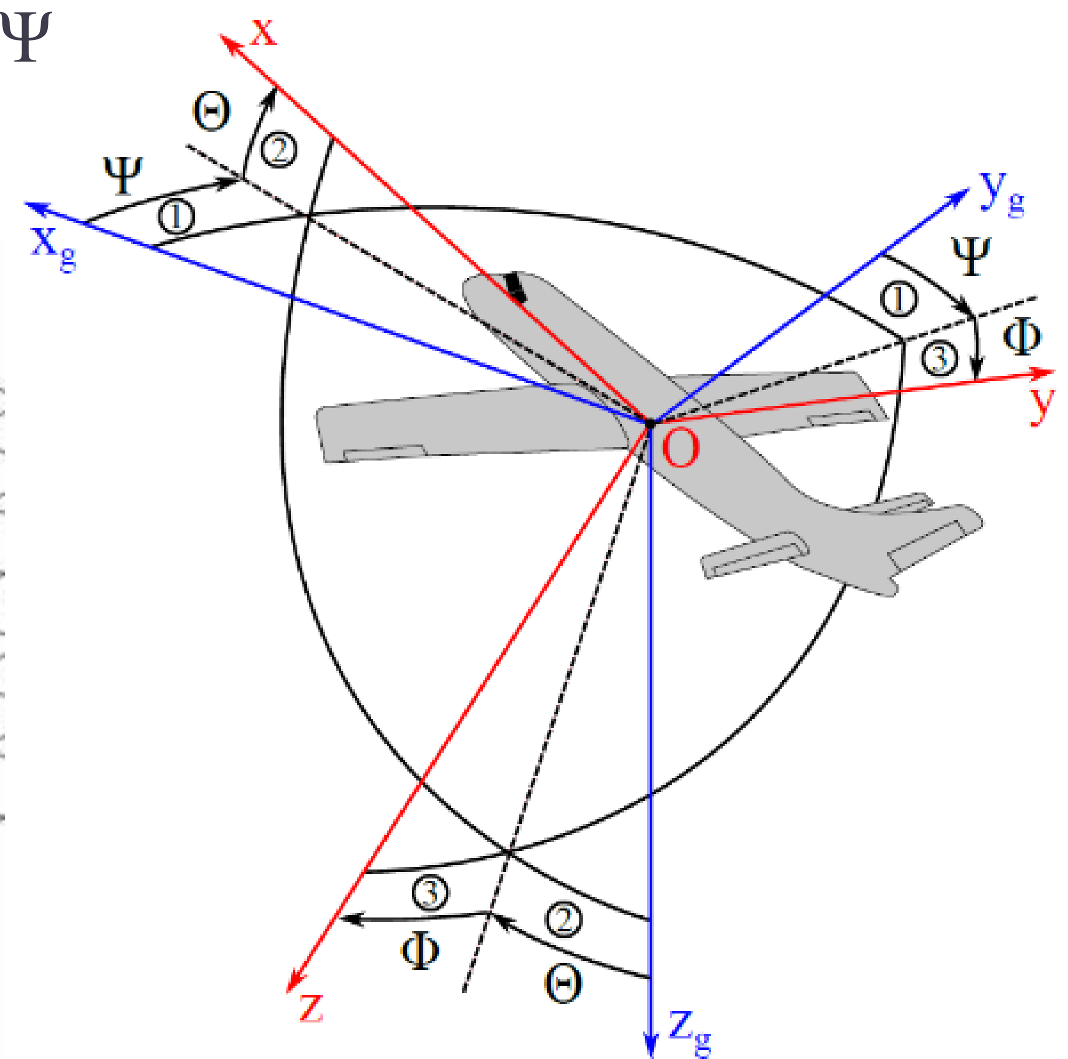
- Body fixed reference frame Oxyz
  - Point O is an arbitrary point of the aircraft (lies in the symmetry plane)
  - Ox axis lies in the Oxz plane and is parallel to mean aerodynamic chord
  - Oy axis is directed towards right wing
  - Oz axis is directed downward and completes the right-handed frame



Attitude

- Tait-Bryant angles (Euler angles convenction for aeronautics)
  - Yaw angle Ψ – angle of rotation along Oz_g axis. After the rotation, Ox_g axis coincide with the projection of the longitudinal axis Ox on the horizontal plane $Ox_g y_g$
  - Pitch angle Θ – Rotation angle of the vertical plane $Oy_g z_g$. After the rotation, the Ox_g axis rotated by Ψ coincides with the longitudinal axis Ox
  - Roll angle Φ – angle of rotation along longitudinal axis. After the rotation, Oy_g axis rotated by Ψ coincides with the Oy axis





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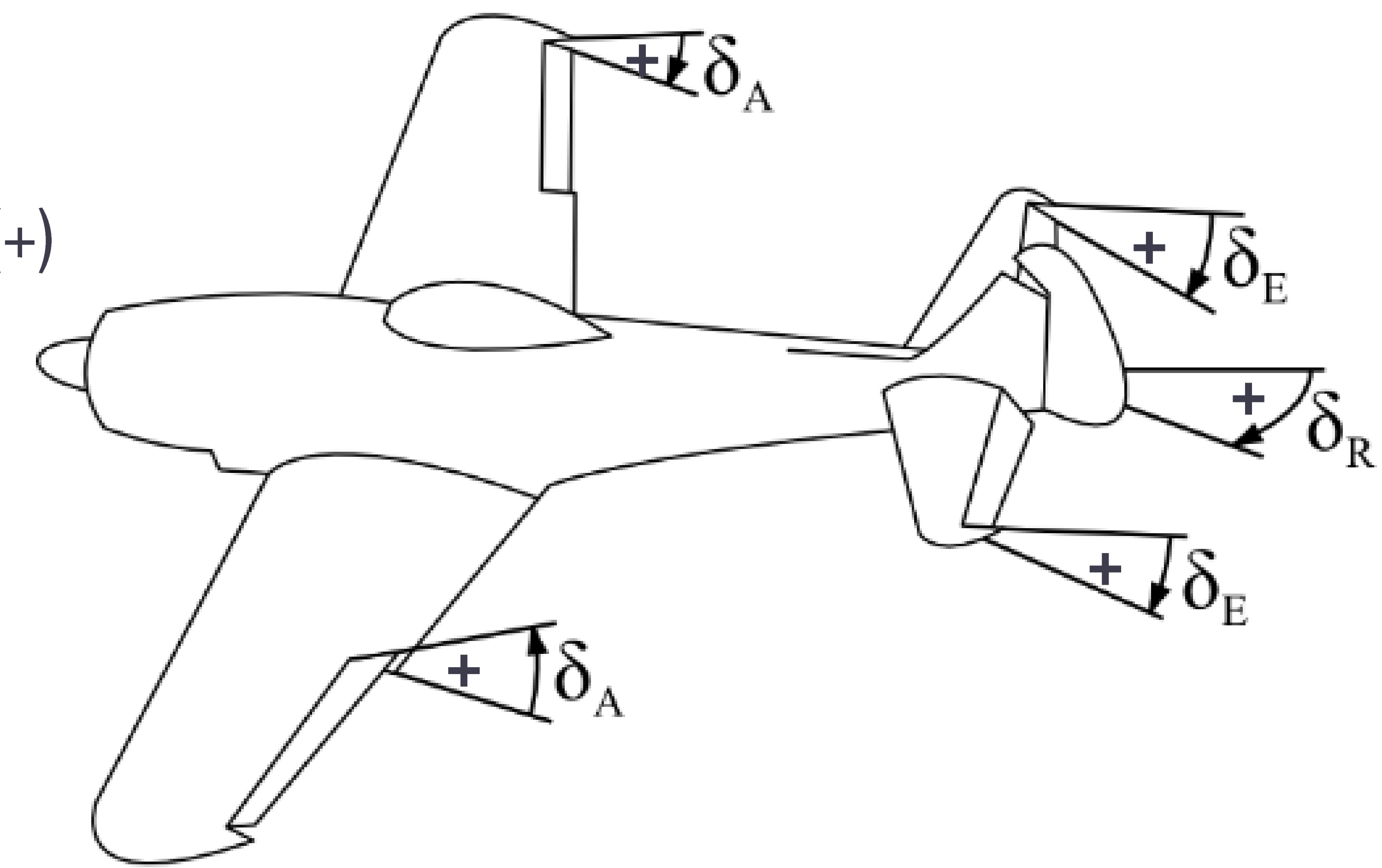
Flight controls deflection

- „Positive" pilot action results in „negative" flight surface deflection and „positive" aircraft reaction (with respect to particular axis)
  - Pitch: pull the stick (+) → elevator up (-) → aircraft nose pitch up (+)
  - Roll: push the stick right (+) → right aileron up, left aileron down (-) → right wing down, left wing up (+)
  - Yaw: push right pedal (+) → rudder right (-) → aircraft nose yaw right (+)

- Engine control
  - Push the thrust lever (+)
    → engine power increased (+)
    → speed increased (+)



Dynamic equations of motion

- Change theorems when origin at CG

$$\frac{d\mathbf{\Pi}}{dt} = \mathbf{F} \qquad \frac{d\mathbf{K}_O}{dt} = \mathbf{M}_O \qquad \frac{d\mathbf{c}}{dt} = \frac{\tilde{\delta}\mathbf{c}}{\tilde{\delta}t} + \mathbf{\Omega} \times \mathbf{c}$$

- Rigid body

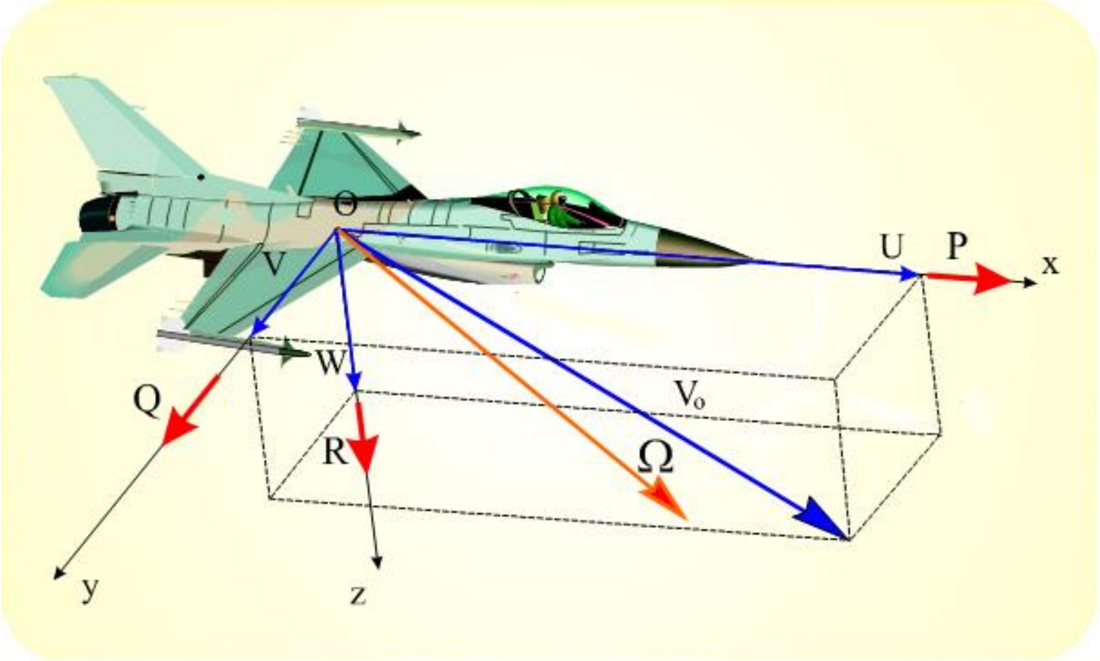
$$\mathbf{\Pi} = m\mathbf{V}_O \qquad \mathbf{K}_O = \mathbf{I\Omega}$$

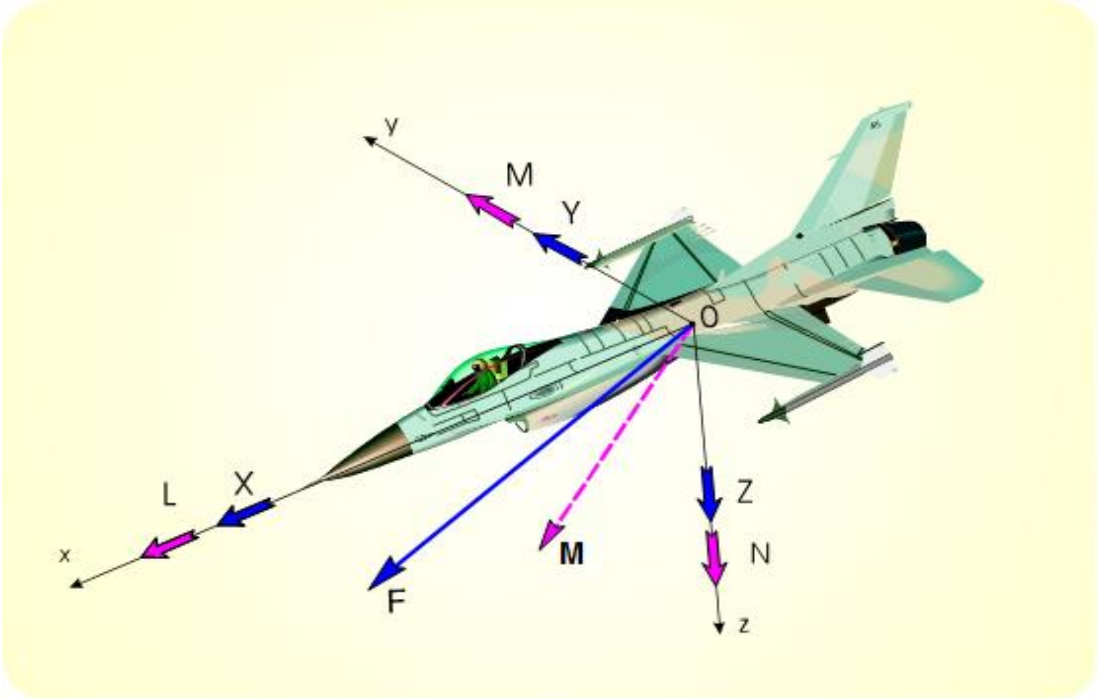
- Symmetry plane

$$\mathbf{I} = \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix}$$

- Kinematic relationships

- Linear equations – small perturbances
  - Steady straight symmetric flight in equilibrium



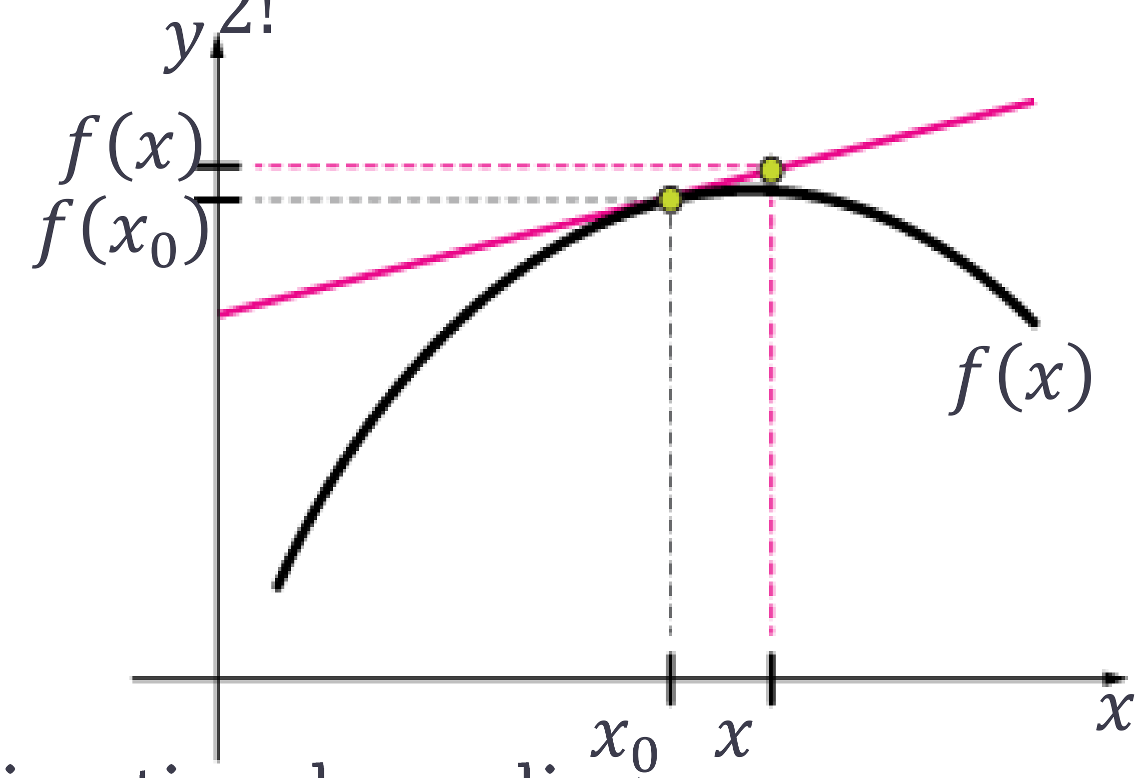


Aerodynamic forces and moments

- Taylor series expansion (usually first order)

$$f(x) = f(x_0) + f^{(1)}(x_0)\frac{(x-x_0)}{1!} + f^{(2)}(x_0)\frac{(x-x_0)^2}{2!} + \cdots + R_n(x, x_0)$$

  - Aerodynamic derivatives valid for a specified flight condition and configuration



- Linear/nonlinear approximation
- Multivariate function
  - State variables
  - Control variables
- Usually ignores longitudinal-lateral-directional coupling

$$X_A = X_{A_0} + \frac{\partial X_A}{\partial U}\Delta U + \frac{\partial X_A}{\partial W}\Delta W + \frac{\partial X_A}{\partial Q}\Delta Q + \frac{\partial X_A}{\partial \delta_E}\Delta\delta_E + \frac{\partial X_A}{\partial \delta_T}\Delta\delta_T + \cdots$$

$$Y_A = Y_{A_0} + \frac{\partial Y_A}{\partial V}\Delta V + \frac{\partial Y_A}{\partial P}\Delta P + \frac{\partial Y_A}{\partial R}\Delta R + + \frac{\partial Y_A}{\partial \delta_A}\Delta\delta_A + \frac{\partial Y_A}{\partial \delta_R}\Delta\delta_R + \cdots$$

Aerodynamic derivatives

- Dimensional

$$X_{\xi_i} = \frac{1}{m}\frac{\partial X}{\partial \xi_i} \qquad Y_{\xi_i} = \frac{1}{m}\frac{\partial Y}{\partial \xi_i} \qquad Z_{\xi_i} = \frac{1}{m}\frac{\partial Z}{\partial \xi_i}$$

$$L_{\xi_i} = \frac{1}{I_{xx}}\frac{\partial L}{\partial \xi_i} \qquad M_{\xi_i} = \frac{1}{I_{yy}}\frac{\partial M}{\partial \xi_i} \qquad N_{\xi_i} = \frac{1}{I_{zz}}\frac{\partial N}{\partial \xi_i}$$

- Non-dimensional

$$C_X = \frac{X}{\frac{1}{2}\rho V_0^2 S} \qquad C_Y = \frac{Y}{\frac{1}{2}\rho V_0^2 S} \qquad C_Z = \frac{Z}{\frac{1}{2}\rho V_0^2 S}$$

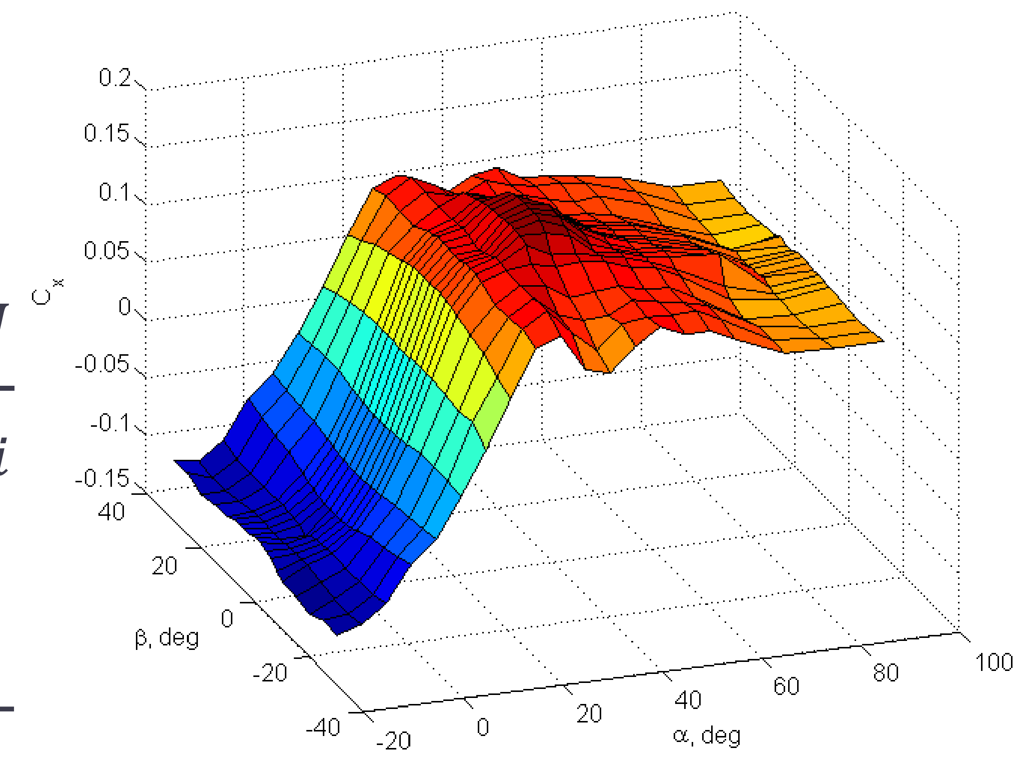
$$C_l = \frac{L}{\frac{1}{2}\rho V_0^2 Sb} \qquad C_m = \frac{M}{\frac{1}{2}\rho V_0^2 S\bar{c}} \qquad C_n = \frac{N}{\frac{1}{2}\rho V_0^2 Sb}$$

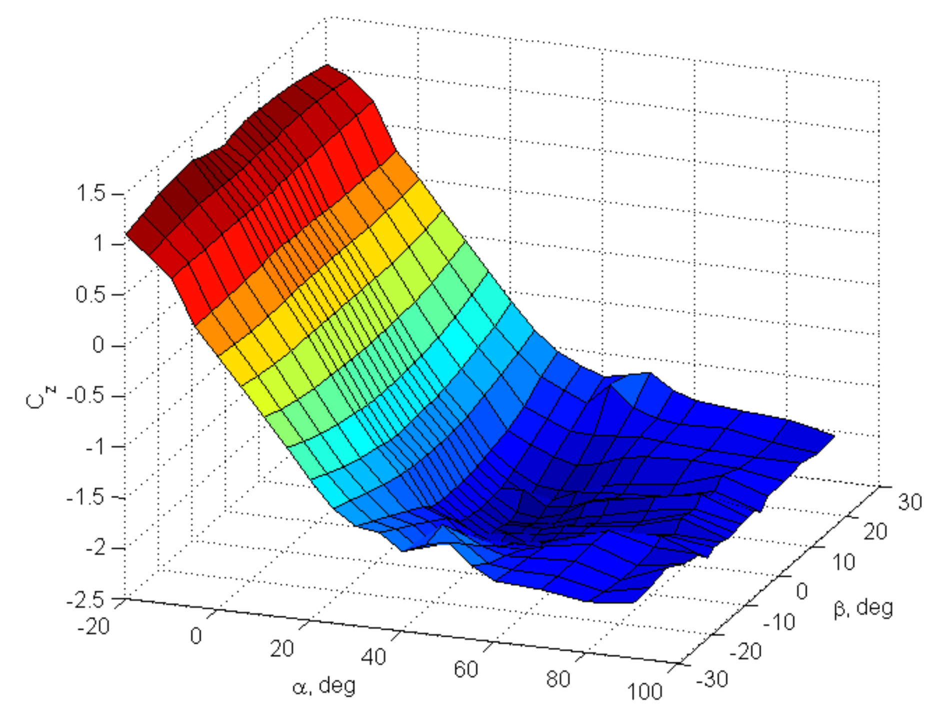
  - for a given flight parameter

$$C_{j_{\xi_i}} = \frac{\partial C_j}{\partial \xi_i^*} \qquad u^* = \frac{\Delta U}{V_0} \qquad w^* = \frac{\Delta W}{V_0} \qquad q^* = \frac{\Delta Q\bar{c}}{2V_0}$$

$$v^* = \frac{\Delta V}{V_0} \qquad p^* = \frac{\Delta Pb}{2V_0} \qquad r^* = \frac{\Delta Rb}{2V_0}$$

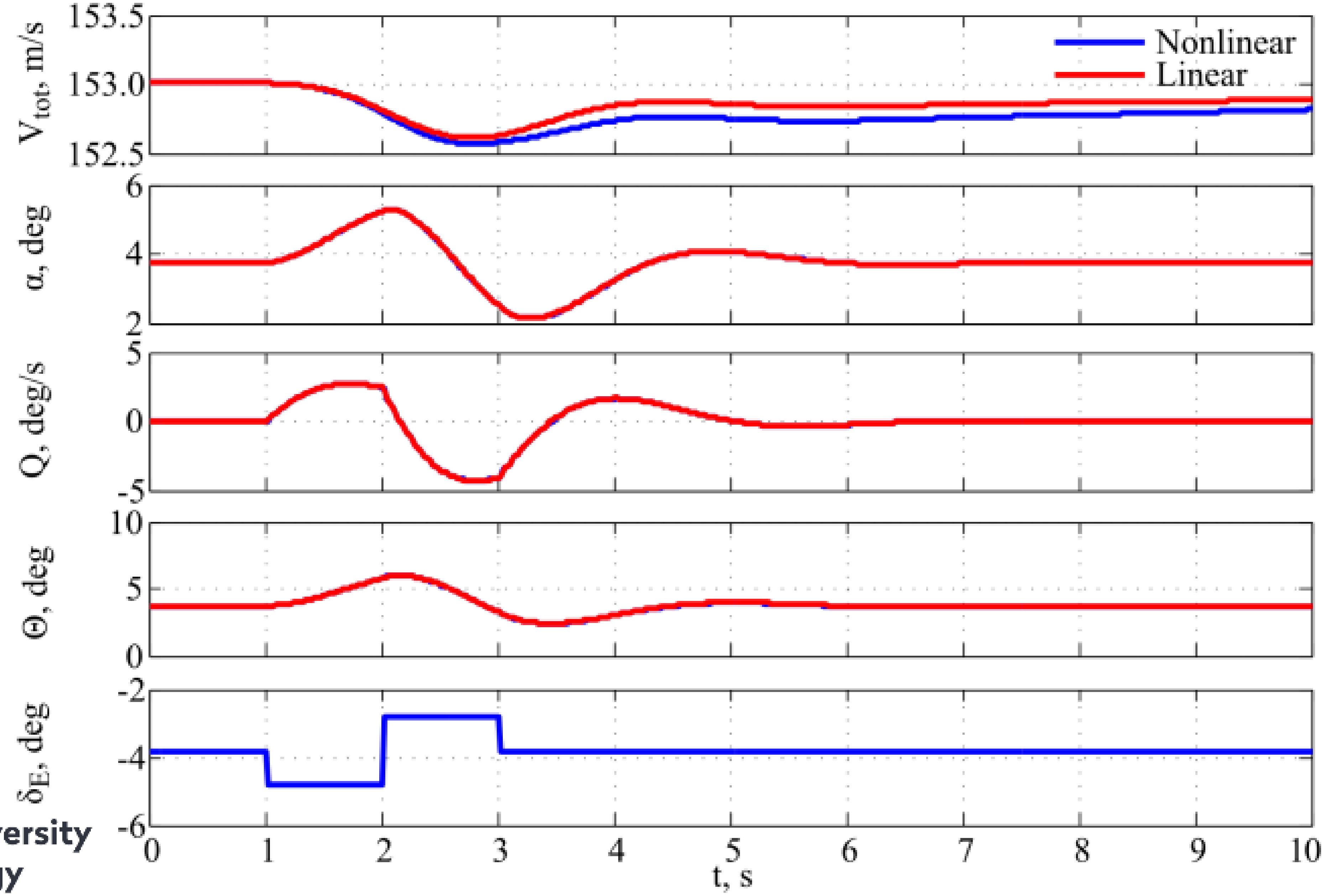
$$C_X = C_{X_0} + C_{X_u}u^* + C_{X_w}w^* + C_{X_q}q^* + C_{X_{\delta_E}}\delta_E + C_{X_{\delta_T}}\delta_T + \cdots$$





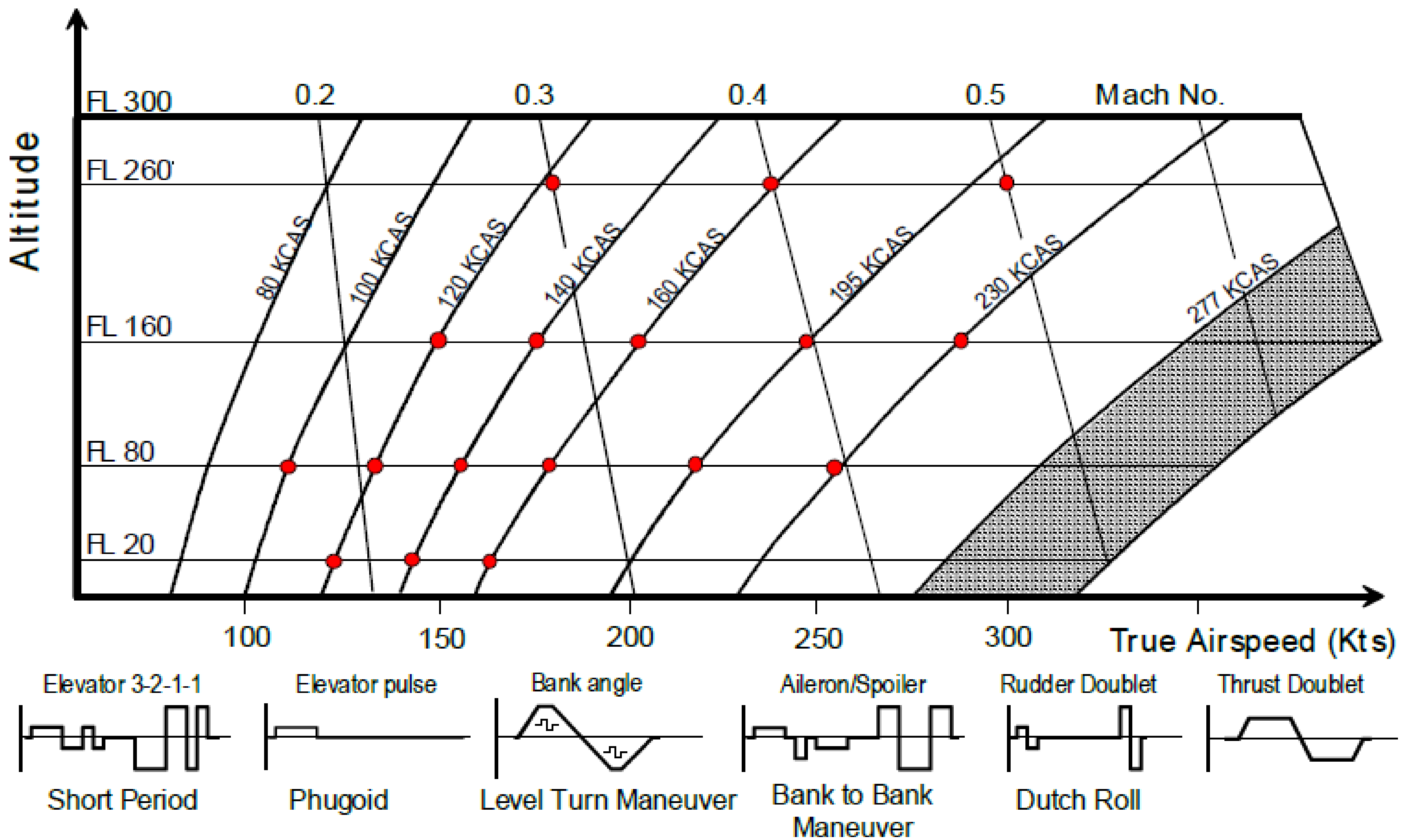
Nonlinear vs linear

$$\begin{bmatrix}\Delta\dot{V}_O\\ \Delta\dot{\alpha}\\ \Delta\dot{Q}\\ \Delta\dot{\Theta}\end{bmatrix}=\begin{bmatrix}-0.0171 & -3.6619 & -1.0969 & -32.174\\ -0.003 & -0.7534 & 0.9279 & 0\\ 0 & -4.3115 & -1.2657 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta V_O\\ \Delta\alpha\\ \Delta Q\\ \Delta\Theta\end{bmatrix}+\begin{bmatrix}0.0999\\ -0.0016\\ -0.1397\\ 0\end{bmatrix}\Delta\delta_E$$



Experiment

- Registered signals determine scope and accuracy of the estimated parameters
  - Determine flight test aim
  - Select aircraft configuration
  - Select trim points
  - Select manoeuvres and specify the excitations



Experiment

- Procedure for each maneuver
  - Flight at specified height at specified velocity
  - Determine aircraft trim configuration
  - Deflect flight control and perform a maneuver
  - Return to the equilibrium (steady horizontal flight)
  - The same maneuver is performed multiple times to eliminate disturbances

Flights	Task	Hours
1	Check flight test instrumentation	2,0
1	Envelope expansion	1,5
2	Airdata system calibration	8,5
14	System identification and model validation (4 altitudes, 5 speeds, 37 configurations)	42,5
2	Ground effects identification	4,5
1	22 stall maneuvers with 5 configurations	3,0
2	Ground and taxi tests	3,5
4	Noise recording in hangar, on runway and in flight	7,5
3	Special tests: load drop, takeoff and landing on unprepared runway and runway with snow	5,0
30	~1000 maneuvers and 37 configurations	78,0

Frequency sweep (chirp signal)

- Continuous sinusoidal signal with frequencies that have to cover range of interest
- Usually used for helicopters and STOL/VTOL aircrafts Sys-ID

- Starts with low frequencies, from 0.1 rad/s to 10 rad/s
- Long duration 60-90s
- The exact shape is of secondary importance
  - Often faded at the beginning
- Only one flight control can be deflected at a time
  - Other controls are used to suppress additional motions
- Problem with staying at certain flight condition
- Can lead to resonance – flight safety
- Used when there is no or small amount of knowledge about the system

Frequency sweep

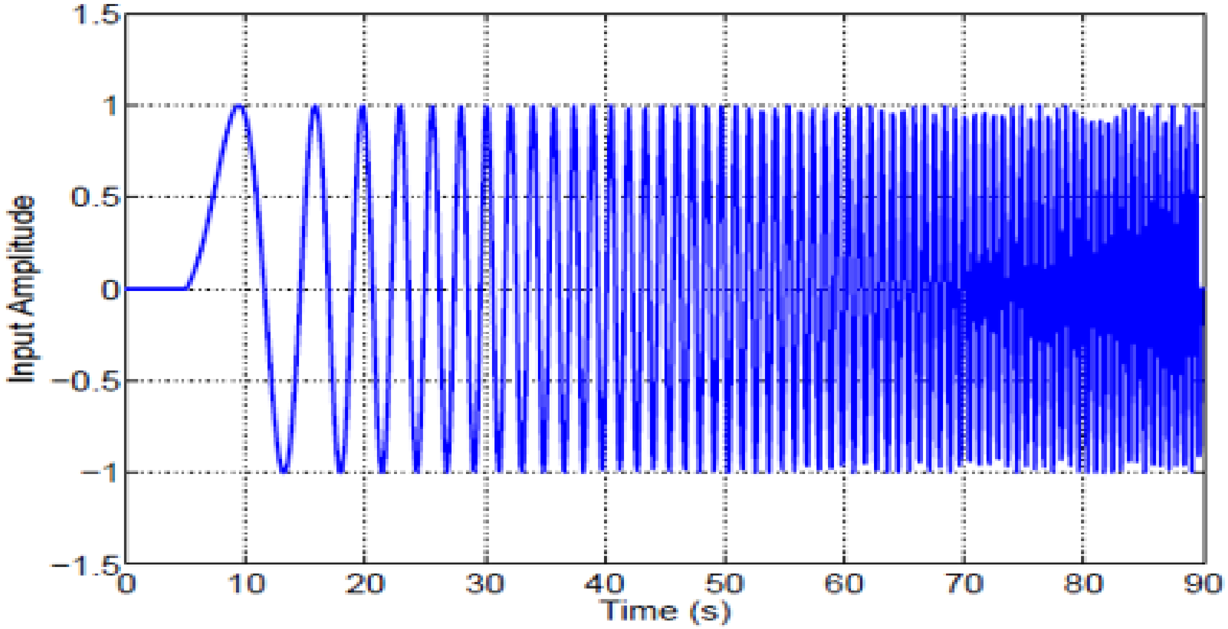
Linear

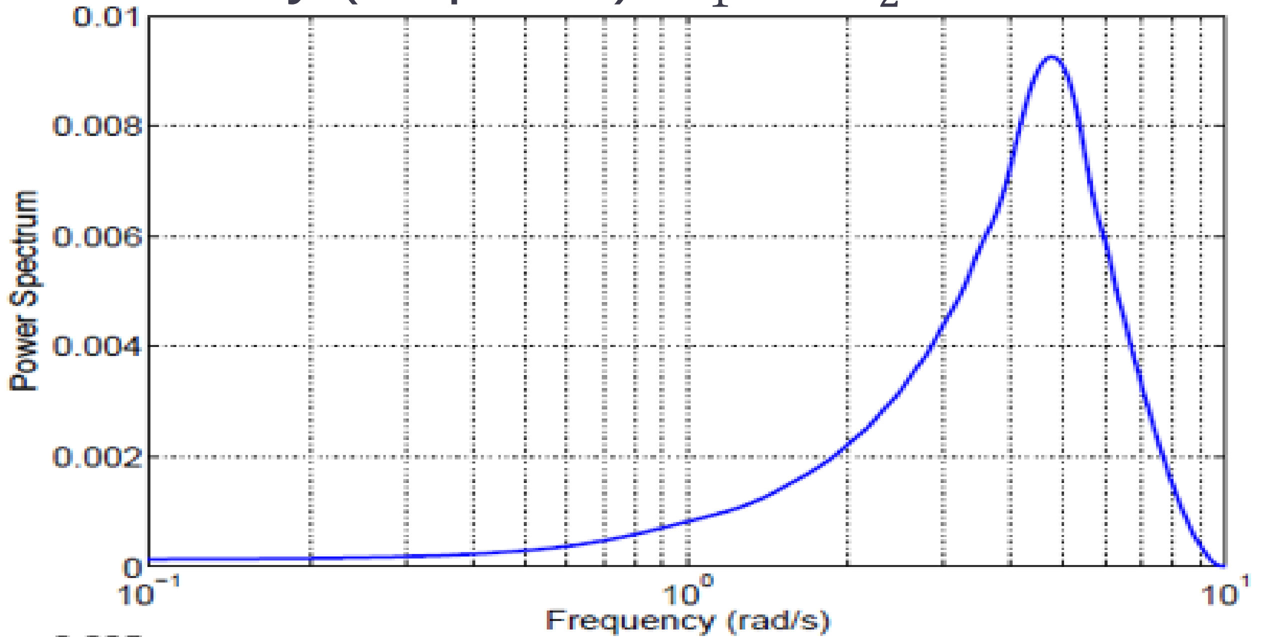
$$u = \sin\left(\omega_0 t + \frac{1}{2}(\omega_1 - \omega_0)\frac{t^2}{T}\right)$$

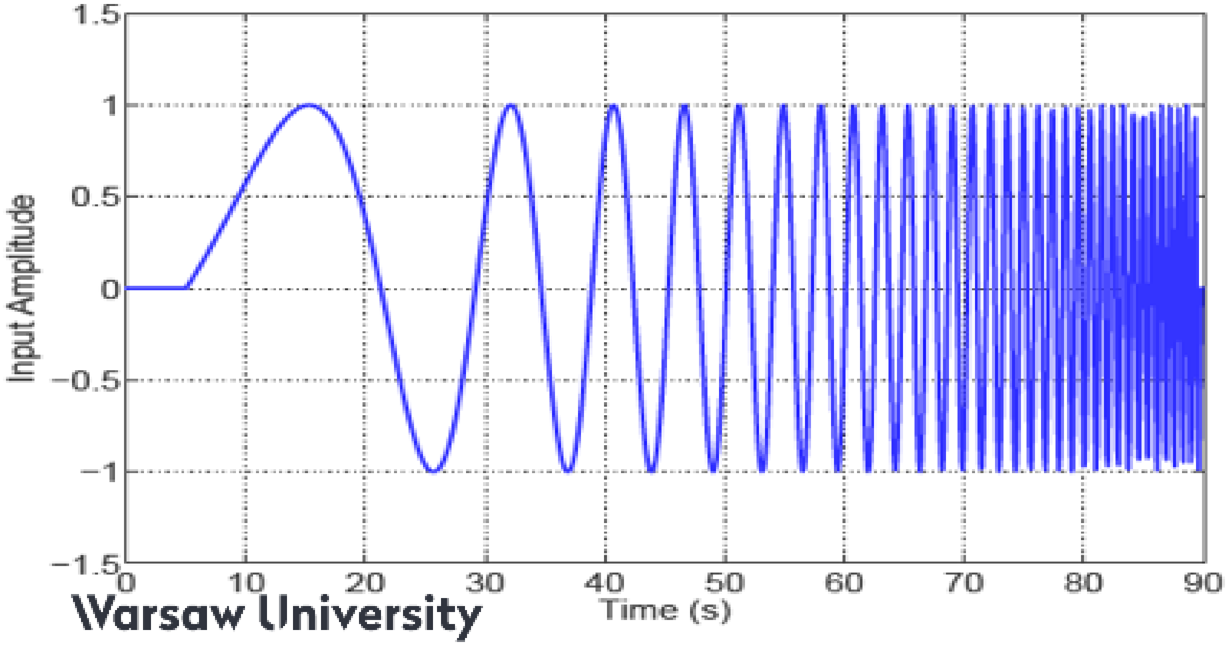
Logarithmic

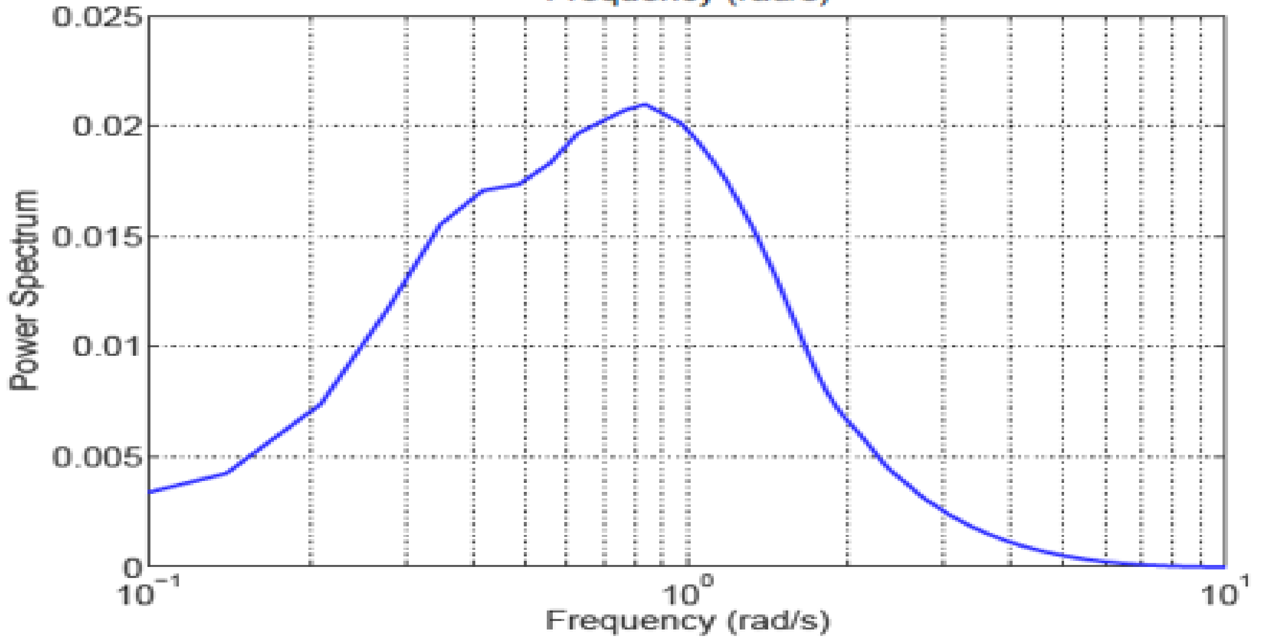
$$u = \sin\left(\omega_0 t + C_1(\omega_1 - \omega_0)\left(\frac{T}{C_2}e^{C_1 t/T} - t\right)\right)$$

Usually (empirical): $C_1=4$, $C_2=0.01876$





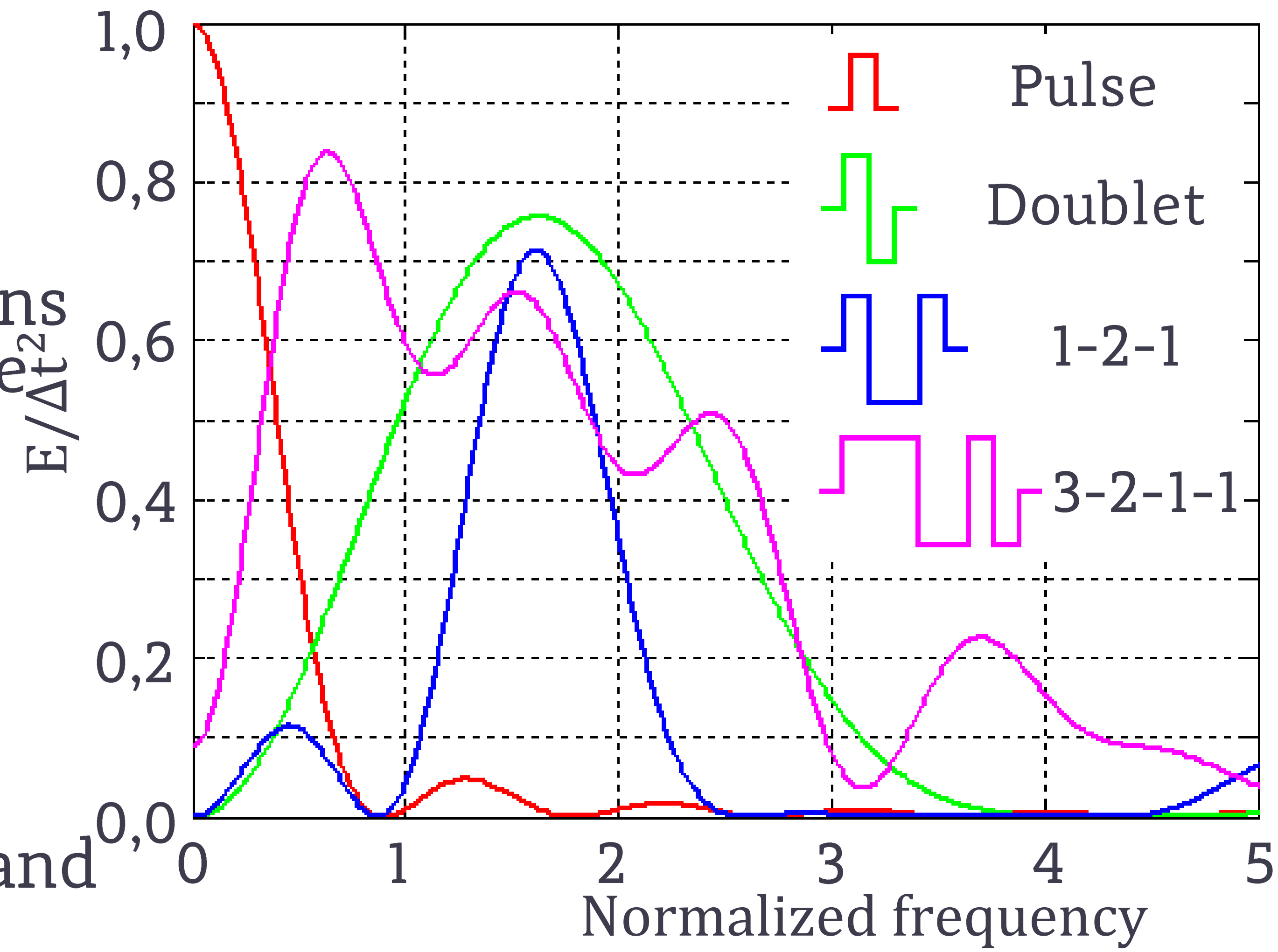




Multi-step - Pulse

Multi-step signals:

- Easy to apply
- Require a-priori model
- Abrupt flight control deflections can cause high loads and induce aeroelastic phenomena



Pulse - the simplest multi-step

- Energy located in a narrow frequency band
  - Broadening the frequency band causes a decrease in energy - it may be too low to excite the system
- Can only be used for exciting low frequency motion (e.g. phugoid)
- Asymmetric signal (non-zero energy for zero frequency)
- Switching time Δt selected for a selected frequency

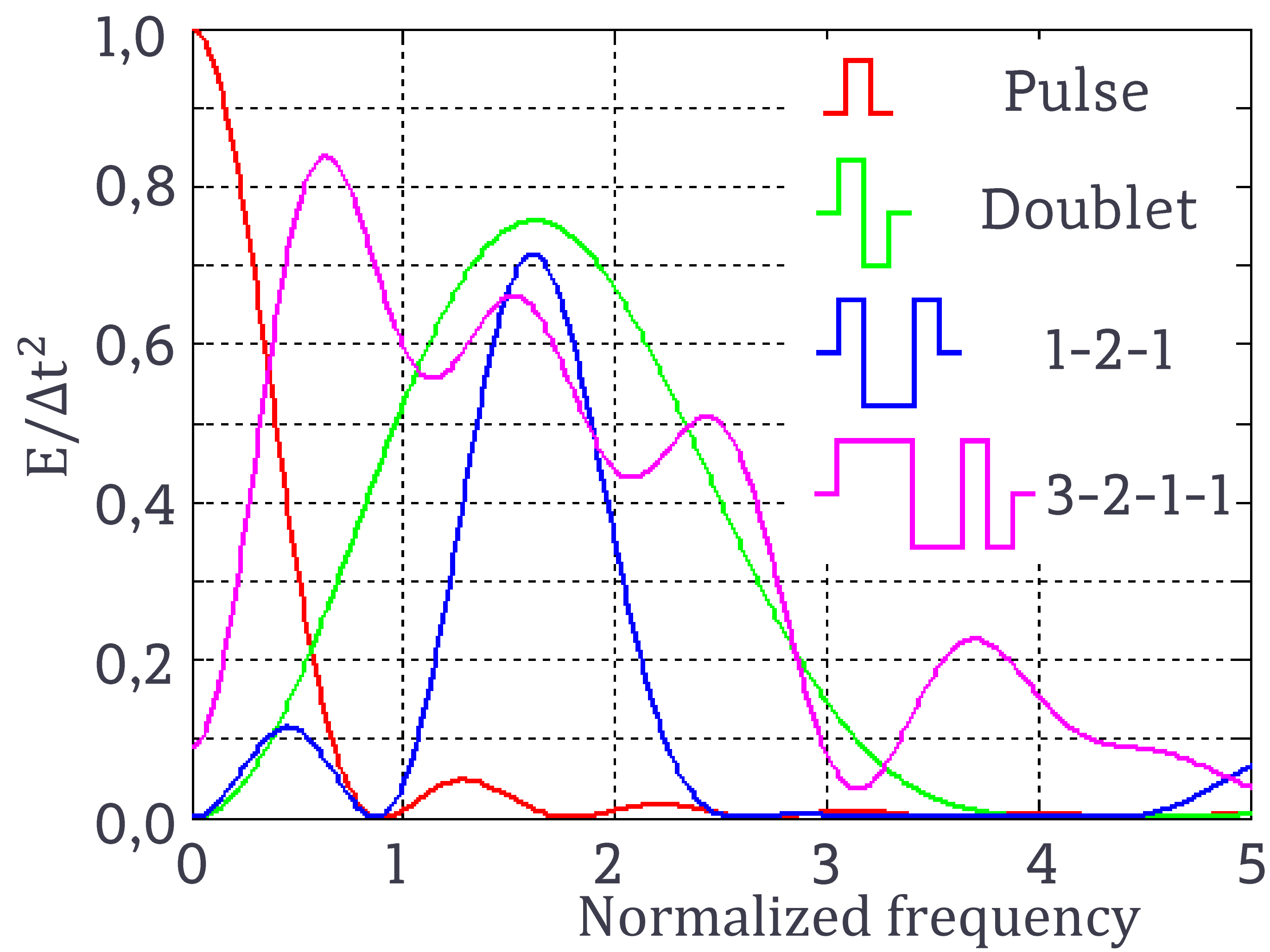
Multi-step - Doublet

Doublet - Composition of two pulse signals

- Much wider frequency band: 50% of the signal energy in a bandwidth 1:3
- Symmetrical signal
- Switching time

  $\omega\Delta t \approx 2.3$

  $$\Delta t \approx \frac{2.3}{\omega} \approx \frac{2\pi}{2.7\omega} \approx \frac{1}{2.7}T$$

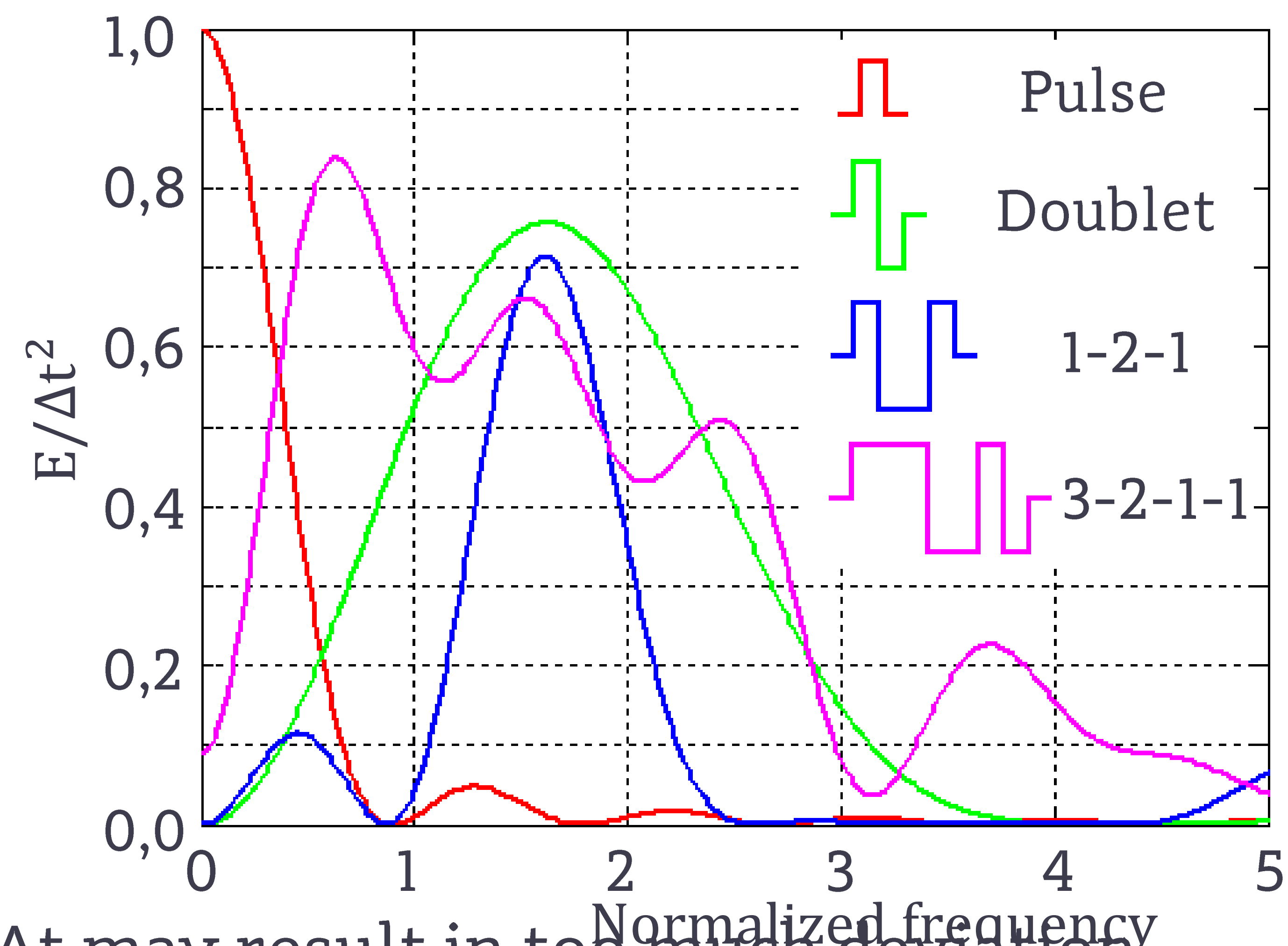


- In practice, Δt is selected to correspond to half the oscillation period T, i.e., the total length of the signal corresponds to the period of motion

Multi-step – 3-2-1-1

3-2-1-1: Composition of pulses with durations in the ratio 3:2:1:1

- Much wider frequency band: 50% of the signal energy in a bandwith 1:10
- Asymmetric signal
- Switching time
  $\omega\Delta t = 1.6 \Rightarrow \Delta t \approx \frac{1}{4}T$
  - often Δt is based on
    $\omega\Delta t = 2.1 \Rightarrow \Delta t \approx \frac{1}{3}T$



- The initial deflection at time $3\Delta t$ may result in too much deviation from the trim point
  - Balance amplitudes
  - 1-1-2-3

1-2-1: Ease in energy spectrum shifting

Switching time selection

- Marchand Method used to determine switching times for multi-steps

1. Build a priori model
2. Obtain Bode magnitude plot
3. Determine frequency bands in which aerodynamic derivatives are identifiable
   - Aerodynamic derivative is identifiable if its magnitude is large in comparison to other derivatives i.e. aerodynamic derivative can not be estimated in a frequency band in which its magnitude is small in comparison to magnitudes of other aerodynamic derivatives
     - Aerodynamic derivative is identifiable if its term has a magnitude of at least 10% of the largest term's magnitude
   - If the magnitude of the inertia term is small in comparison to other terms then aerodynamic derivatives can be estimated only as ratio of themselves
4. Determine switching time Δt

Marchand Method

1. A-priori model

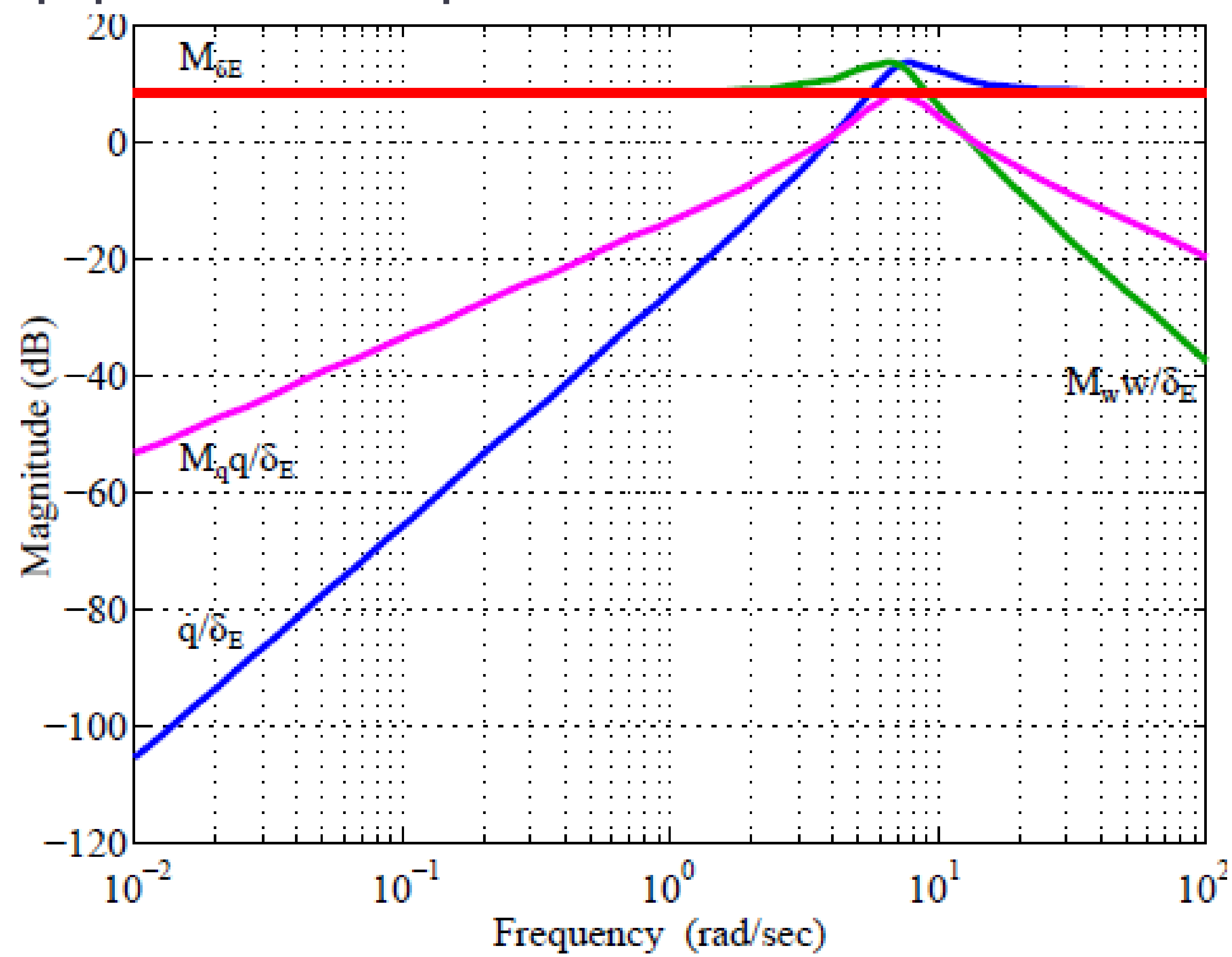
$$\begin{bmatrix}\Delta\dot{W}\\ \Delta\dot{Q}\end{bmatrix} = \begin{bmatrix}0 & -30\\ -1.64 & -4.01\end{bmatrix}\begin{bmatrix}\Delta W\\ \Delta Q\end{bmatrix} + \begin{bmatrix}0\\ -2.61\end{bmatrix}\Delta\delta_E$$

2. Bode plot

$$\left|\frac{\Delta\dot{Q}(\omega)}{\Delta\delta_E(\omega)}\right|, \left|\frac{M_W\Delta W(\omega)}{\Delta\delta_E(\omega)}\right|, \left|\frac{M_Q\Delta Q(\omega)}{\Delta\delta_E(\omega)}\right|, \left|\frac{M_{\delta_E}\Delta\delta_E(\omega)}{\Delta\delta_E(\omega)}\right|$$

3. Frequency range: 6-10 rad/s
   - Selected frequency: 8rad/s
   - Doublet

$$\Delta t \approx \frac{2.3}{\omega} \approx \frac{2.3}{8} \approx 0.3\text{s}$$



Multisine

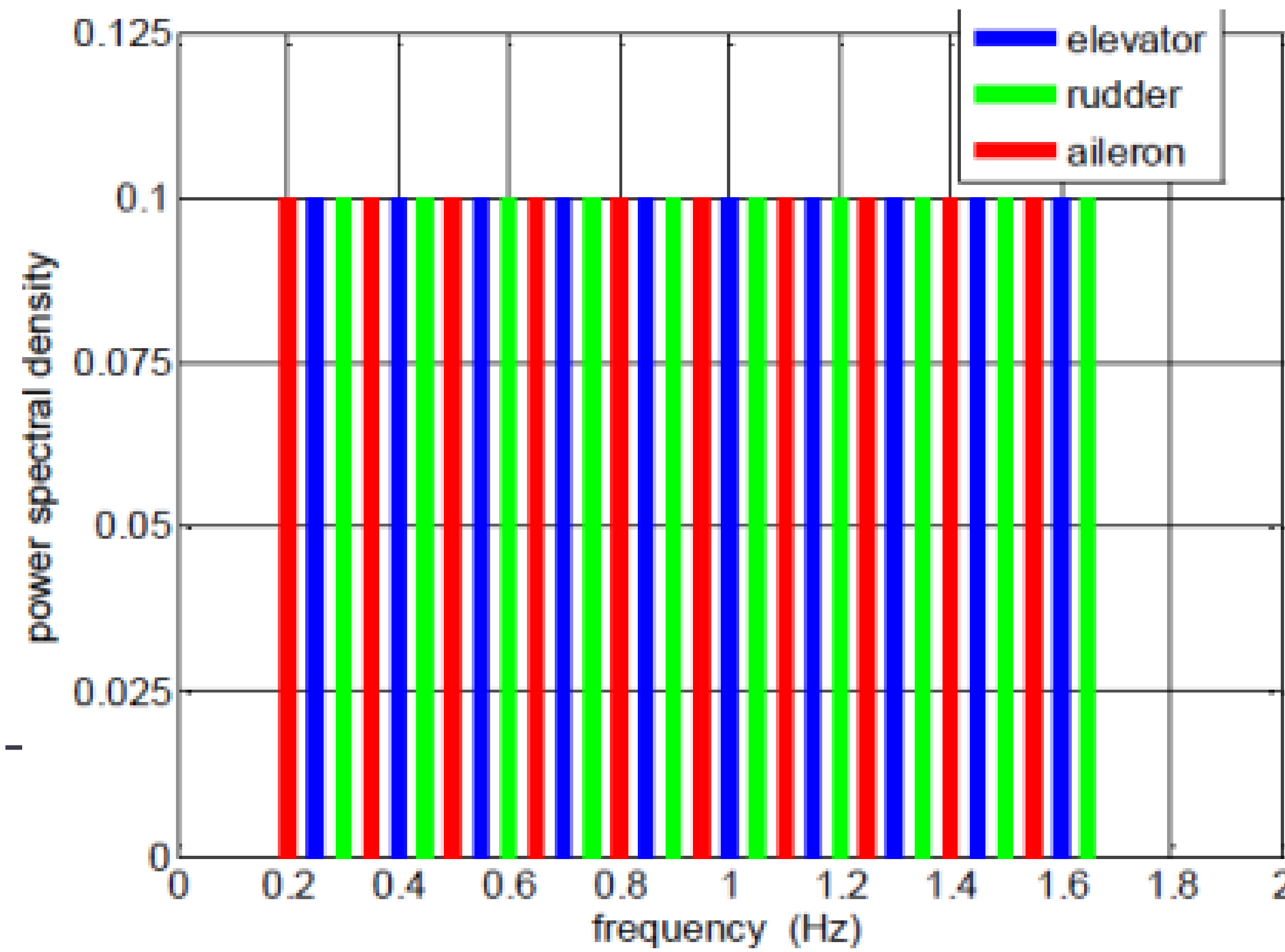
$$u_j = \sum_{k=1}^{M} A_k \sin\left(\frac{2\pi kt}{T} + \varphi_k\right)$$

- Determining the base frequency $f_0=1/T$ and the frequency range $< f_0; Mf_0 >$
- Assigning subsequent harmonics to the control surfaces
- Determination of amplitudes based on the power spectrum
  - Homogeneous spectrum:

    $$A_k = \frac{A_j}{\sqrt{n_j}}$$

  - Non-uniform power spectrum - harmonics

    $$A_k = A_j\sqrt{p_k}$$



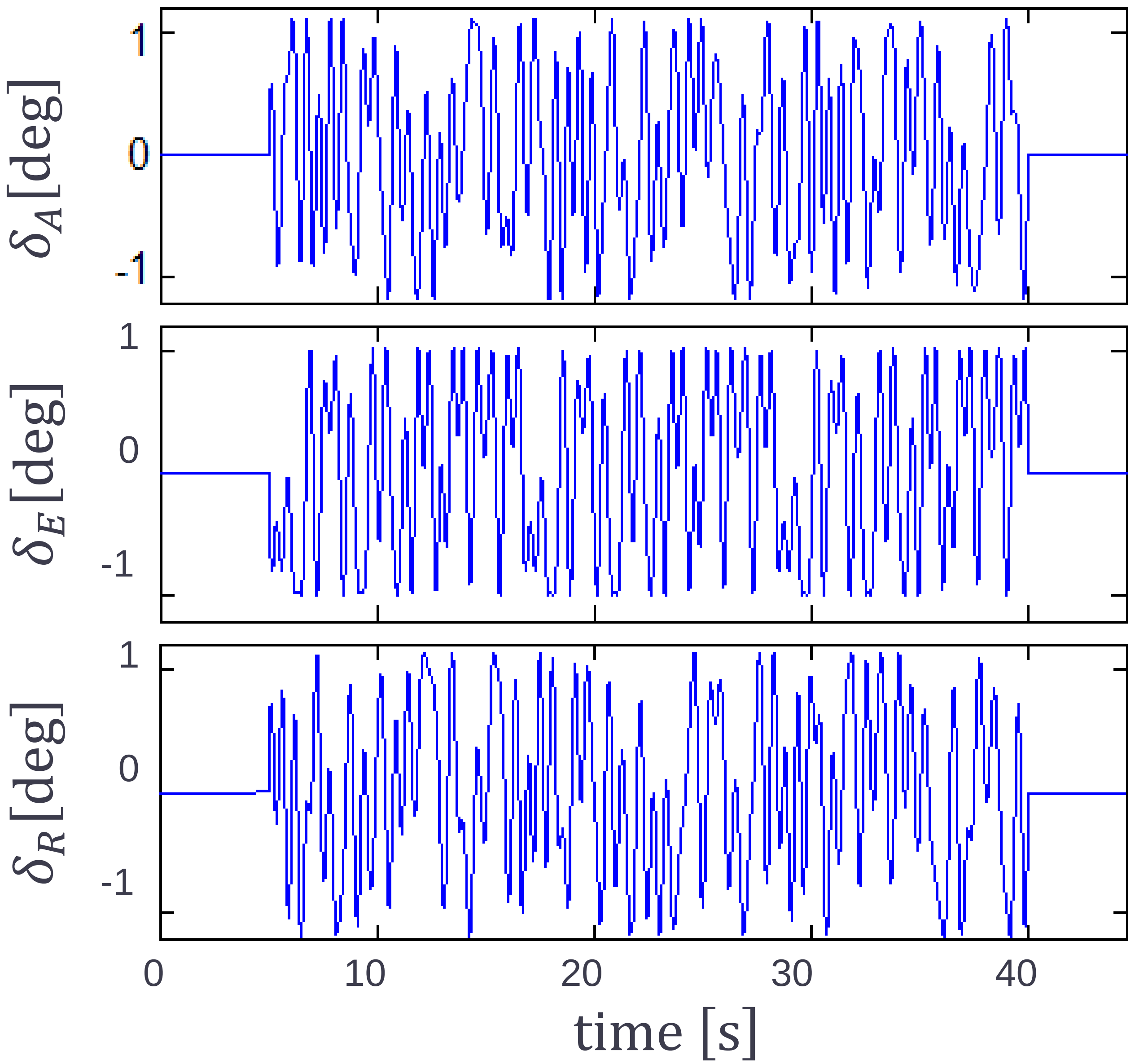
Multisine

$$u_j = \sum_{k=1}^{M} A_k \sin\left(\frac{2\pi kt}{T} + \varphi_k\right)$$

- Signal optimization – selection of phase shift angles φ_k
- Relative Peak Factor RPF - a measure of the effectiveness of the control surface deflection

$$RPF = \frac{\max(u_j) - \min(u_j)}{2\sqrt{2}\,\mathrm{rms}(u_j)}$$

  - Initial values of phase shifts
    - Schroeder's formula



- Shifting the signals along the time axis so that the flight control deflections start and end at zero

Optimallity criteria

- Fisher Information Matrix – measures the amount of information that observable random variables carry about unknown parameters that describe the system

$$\mathbf{F} \equiv \mathrm{E}\left\{\left[\frac{\partial \ln L(\mathbf{\Theta}|\mathbf{z})}{\partial \mathbf{\Theta}}\right]\left[\frac{\partial \ln L(\mathbf{\Theta}|\mathbf{z})}{\partial \mathbf{\Theta}}\right]^{\mathrm{T}}\right\} = \mathrm{E}\left[-\frac{\partial^2 \ln L(\mathbf{\Theta}|\mathbf{z})}{\partial \mathbf{\Theta}\partial \mathbf{\Theta}^{\mathrm{T}}}\right]$$

- Likelihood function

$$L(\mathbf{\Theta}|\mathbf{z}) \equiv p(\mathbf{z}|\mathbf{\Theta}) \qquad \int_{-\infty}^{+\infty} p(\mathbf{z}|\mathbf{\Theta})dz = 1$$

  - Multivariate normal distribution (for all time points)

$$p(\mathbf{z}|\mathbf{\Theta}) = \left((2\pi)^{n_y}|\mathbf{R}|\right)^{-N/2} \exp\left(-\frac{1}{2}\sum_{k=1}^{N}[\mathbf{z}(t_k) - \mathbf{y}(t_k)]^{\mathrm{T}}\mathbf{R}^{-1}[\mathbf{z}(t_k) - \mathbf{y}(t_k)]\right)$$

  - Covariance matrix

$$\mathbf{R} = \begin{bmatrix} {\sigma_1}^2 & \sigma_1\sigma_2\rho_{1,2} & \cdots & \sigma_1\sigma_{n_y}\rho_{1,n_y} \\ \sigma_2\sigma_1\rho_{2,1} & {\sigma_2}^2 & \cdots & \sigma_2\sigma_{n_y}\rho_{2,n_y} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n_y}\sigma_1\rho_{n_y,1} & \sigma_{n_y}\sigma_2\rho_{n_y,2} & \cdots & {\sigma_{n_y}}^2 \end{bmatrix}$$

Optimallity criteria

- Fisher Information Matrix sensitivity form

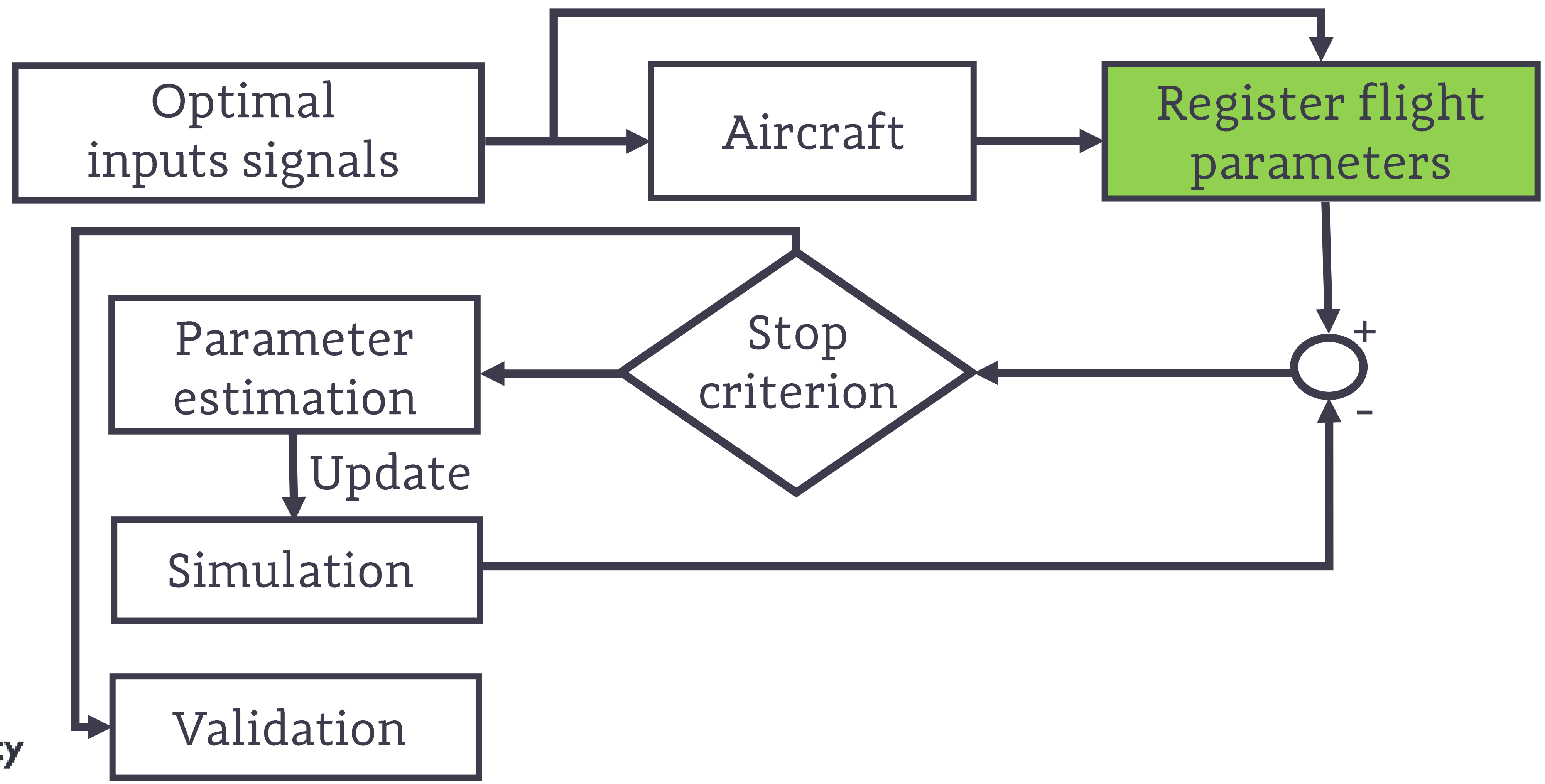
$$\mathbf{F} \approx \sum_{k=1}^{N} \left[\frac{\partial \mathbf{y}(t_k)}{\partial \mathbf{\Theta}}\right]^T \mathbf{R}^{-1} \left[\frac{\partial \mathbf{y}(t_k)}{\partial \mathbf{\Theta}}\right]$$

  - Obtained for the a-priori model and known (initial estimates) model parameters
  - Depends on the model response y
    - Depends on the output signals

- Task- find the control that maximizes F (maximizes information)
  - Estimates uncertainty minimization
- Select the optimality criterion
  - A – optimal – minimizes trace of the Fisher inverse
  - D – maximizes determinant
  - E – maximizes largest eigenvalue
  - L – A-optimal with weights included
  - ...

Measurement

- Linear and angular velocities
- Linear and angular accelerations
- Aerodynamic angles
- Attitude angles
- Flight surfaces deflections
- ...



Measurement

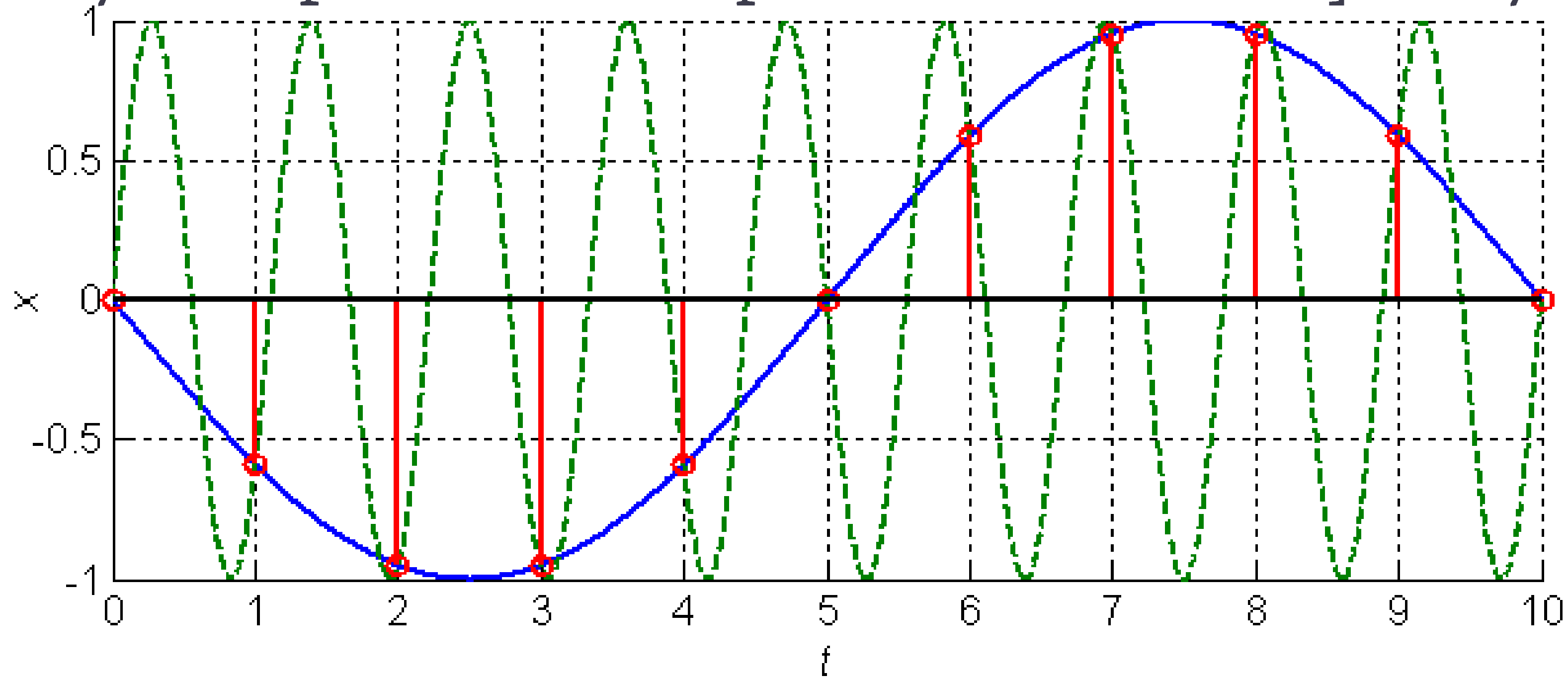
- Whittaker–Nyquist–Kotelnikov–Shannon sampling theorem
  - To faithfully reproduce a continuous signal from discrete signal samples, digital sampling rate f_s must be greater than twice the maximum frequency content f_{max} in the analog signal
    $f_s = 2f_{max}$

    - In practice, sampling at even higher frequency
      $f_s = 25f_{max}$

    - For an scaled object:
      $$f_{model} = \frac{1}{\sqrt{s}} f_{aircraft}$$

Measurement

- The sampled signal contains high frequency components that may be mistakenly interpreted as samples of a lower frequency signal (aliasing)



f_s=1Hz
f_1=0.1Hz
f_2=0.9Hz

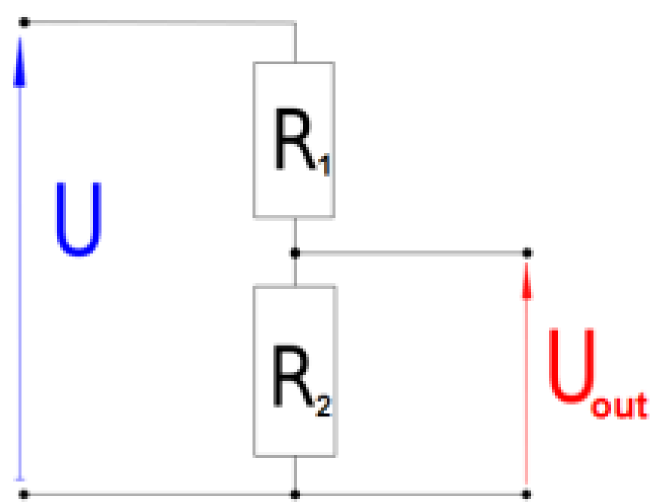
- The need to use anti-aliasing filters before sampling
  - Cut-off frequency f_a less than half f_s
    $f_a \leq f_s/2$
  - In practice, the cut-off frequency is even lower:
    $f_a \leq f_s /5=5\ f_{max}$
- Using filters with the same cut-offs
  - all measured signals have the same time delays

Measurement

- Signals are sampled at 50Hz
- All anti-aliasing filters have the same cut-off frequency
- Raw data recording is recommended
- Flight parameters that are significant in aircraft system identification should
  have the same sampling frequency
  - Flight parameters that are less significant and change slowly in time (e.g. altitude) can have smaller sampling frequency
- If it is possible all signals should be time synchronized
- Signal to noise ratio should be at least 10:1
- Sensors should be calibrated in laboratory and on the object
- Data reduction should be avoided during signals recording
- Data recording process should be observed online in order to check the correctness and completeness of the registered data

Flight controls deflection

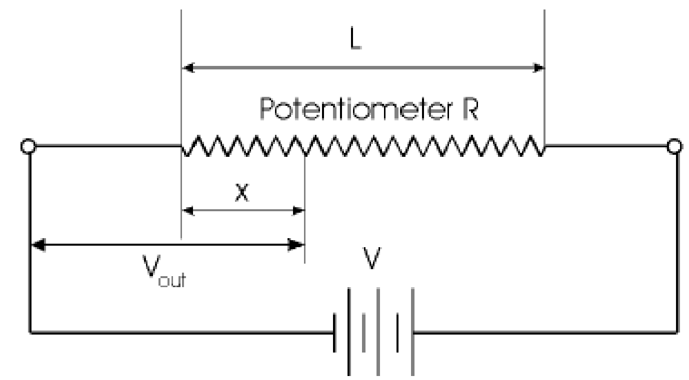
- Potentiometers - adjustable voltage dividers with the ability to change the resistance using the slider
  - Output voltage proportional to displacement or rotation angle



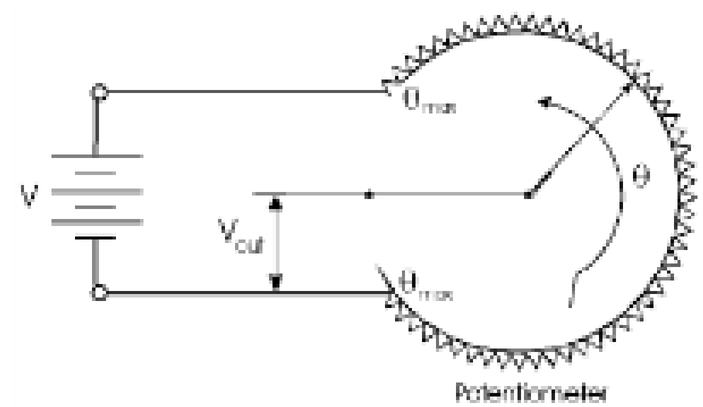
$$U_{out} = U\frac{R_2}{R_1 + R_2}$$

$$R = \rho S l$$

$$R = \rho S \frac{2\pi\, r\theta}{360^o}$$



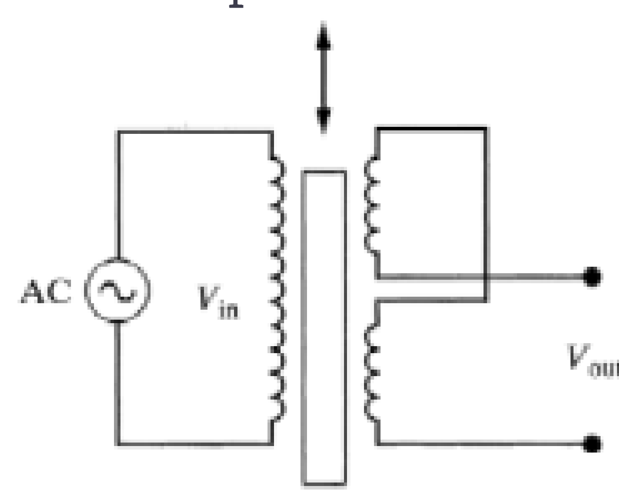
$$U_{out} = U\frac{x}{L}$$



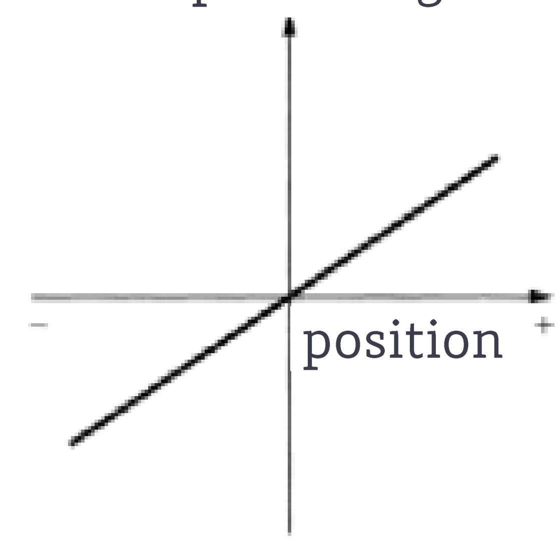
$$U_{out} = U\frac{\theta}{\theta_{max}}$$

- LVDT/RVDT (Linear/Rotational Voltage Differential Transformer) - Changes in the intensity of the magnetic field inside the winding caused by changing the position of the ferrite core

displacement



output voltage



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Accelerometers

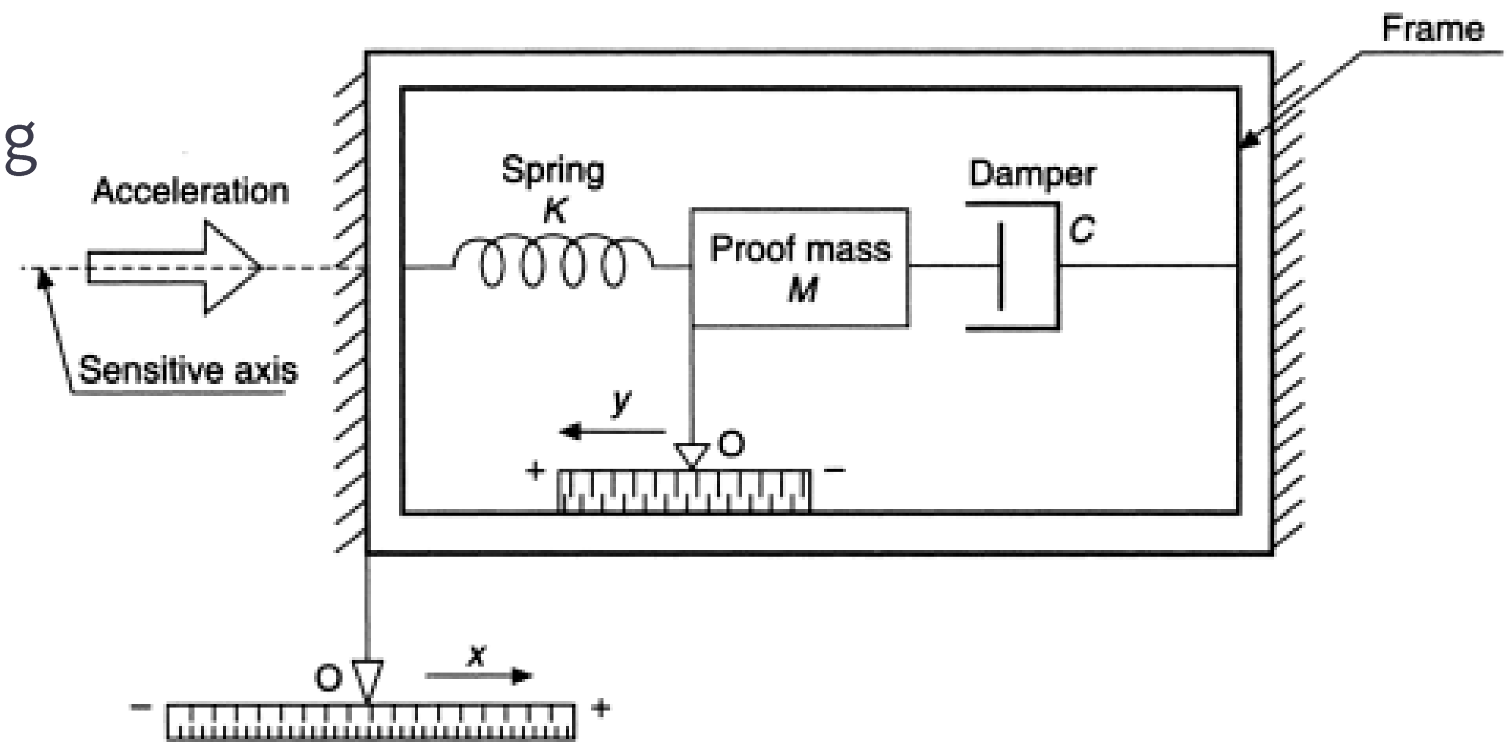
- Inertia force measurement
- Model: Mass – Damper – Spring
  Newton 2nd law of motion, non-inertial frame

$$a = \ddot{x}$$

$$M\ddot{x} = M\ddot{y} + C\dot{y} + Ky$$

steady state: $\ddot{y} = \dot{y} = 0$

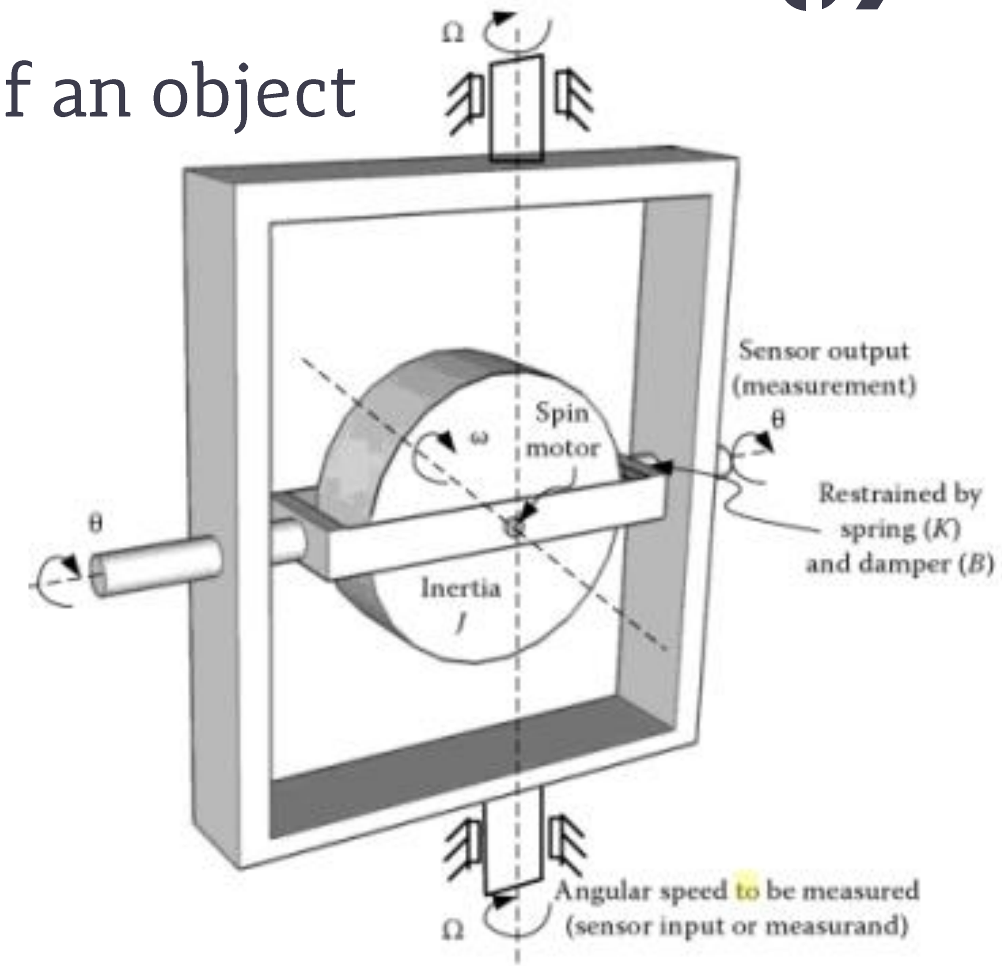
$$a = \frac{K}{m}y$$



- Almost perfectly linear, low bias
- It also records structural and engine responses
  - Need to remove noise from signals

- Tri-axial accelerometer
  - Placed at the center of mass,
  - The sensor axes coincide with the Oxyz system axes
    - Sometimes additional accelerometers are used, e.g. in the cockpit - more SysID possibilities

Angular rates



- Rate gyros - measure the angular velocity of an object
  - The rotating disk can only tilt by θ angle
  - Rotation around the vertical axis causes precession - inclination of the disk by θ (rotation of the spin vector by ΩΔt)

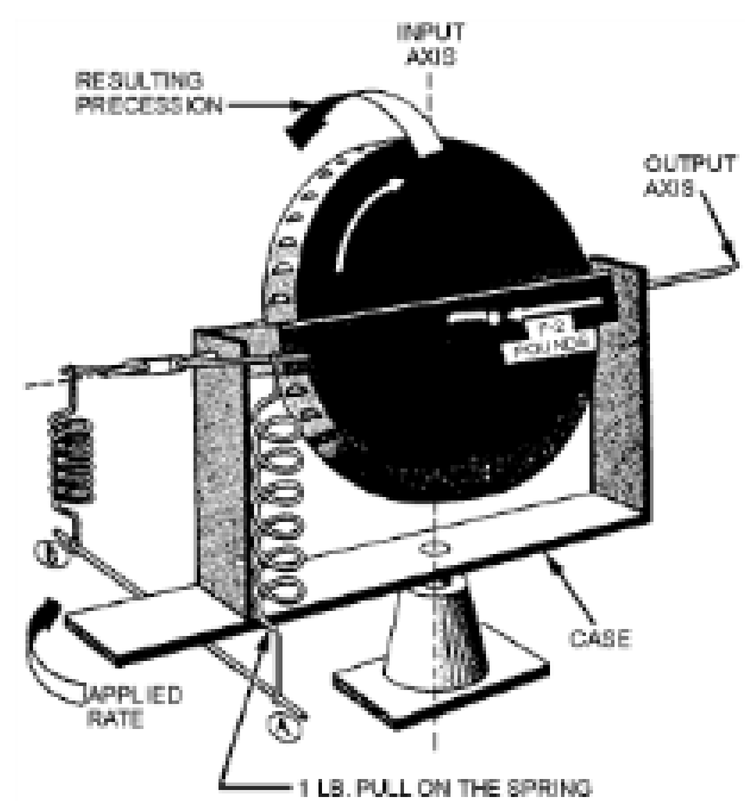
$$H = I\omega$$

$$\Delta H = I\omega\Omega\Delta t \Rightarrow \dot{H} = I\omega\Omega$$

- Springs and damper attached to the horizontal axis

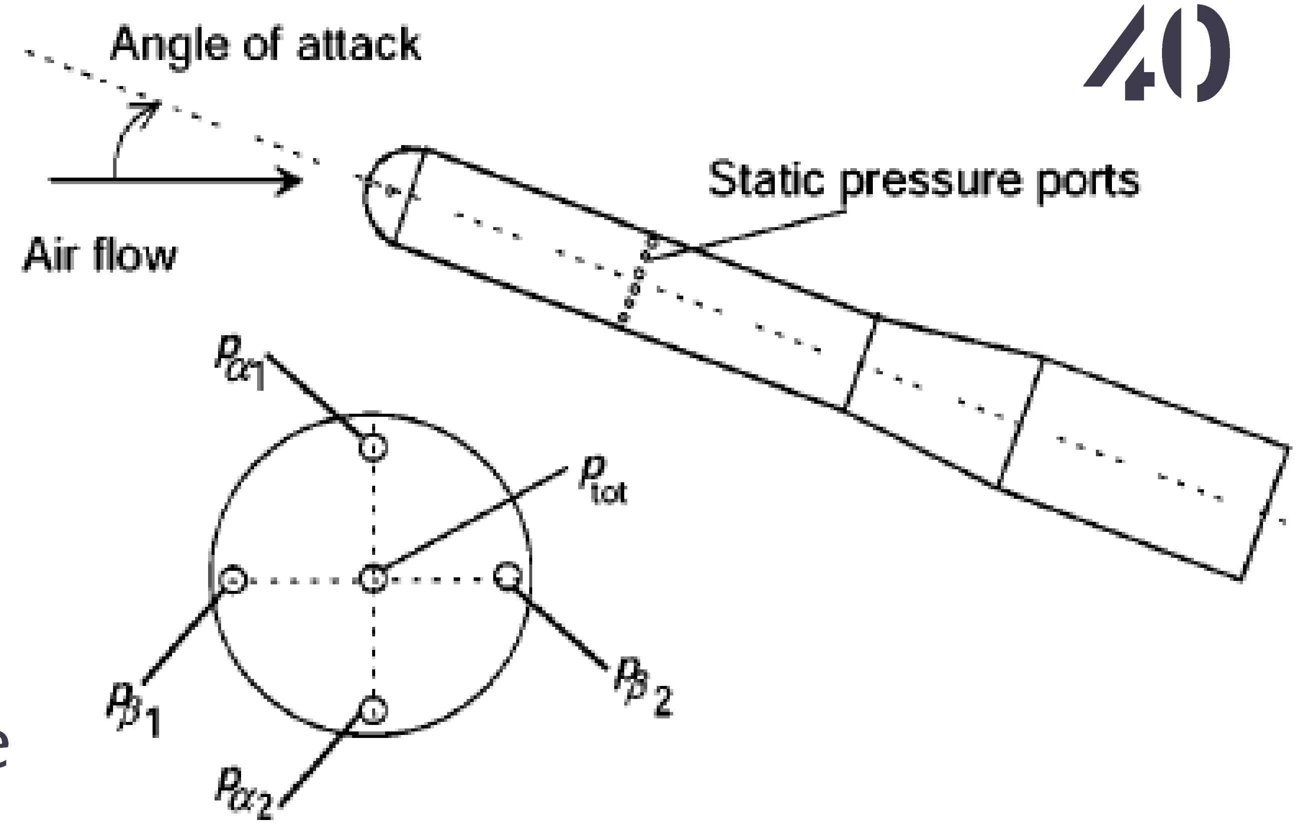
$$\dot{H} = B\dot{\theta} + K\theta \qquad \Omega = \frac{B\dot{\theta} + K\theta}{I\omega} \approx \frac{K\theta}{I\omega}$$

- Almost perfectly linear (B≈0), small bias
- Theoretically, they can be mounted anywhere on the object
  - Practically - the object is not a rigid body and the three gyroscopes are mounted at CG



Aerodynamic angles

- Differential-Pressure Tube
  - Five holes in the head
    - 2 ports for α measurement,
    - 2 ports for β measurement,
    - 1 port for measuring total pressure p_{tot}
  - Side of the tube - static pressure measurement ports p_{st}



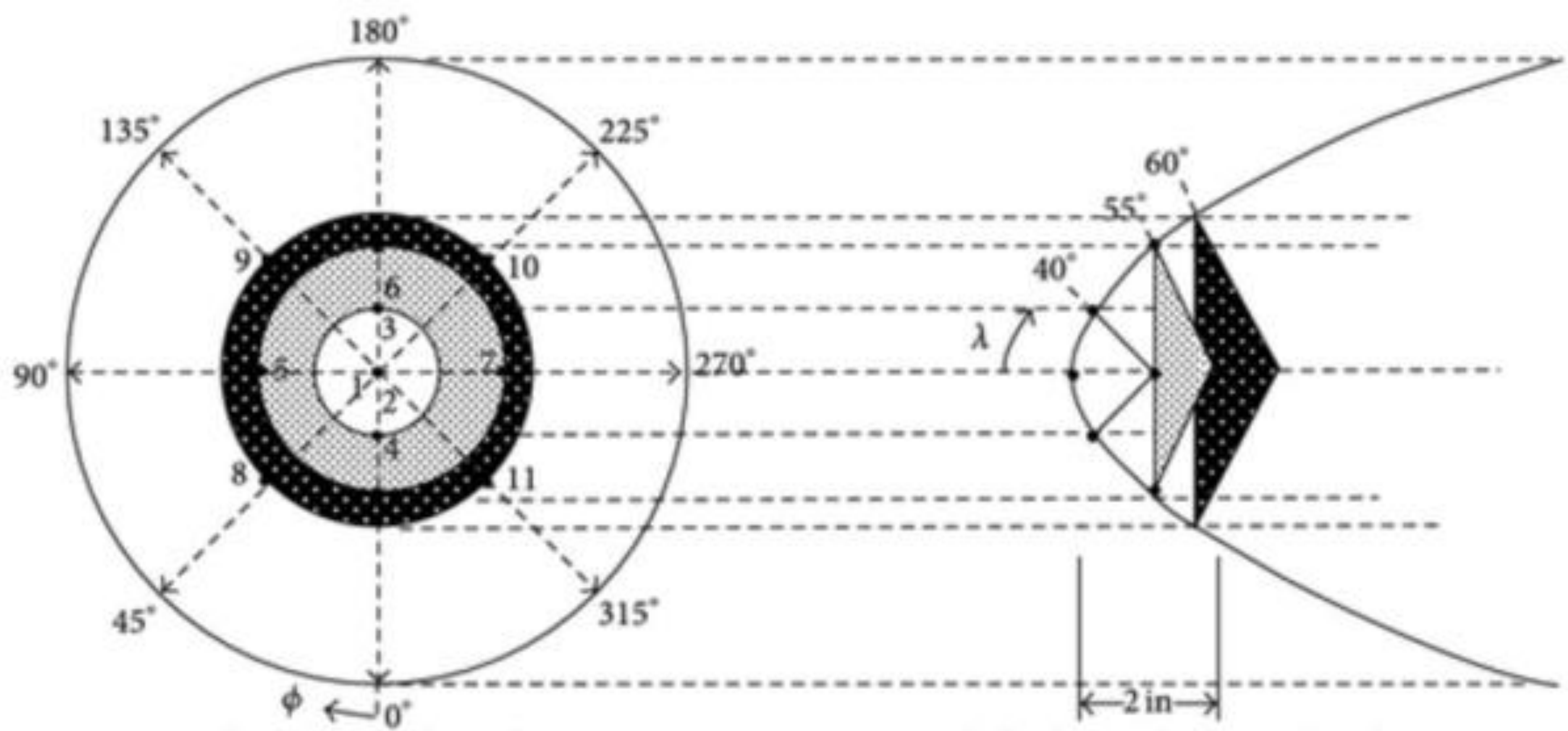
$$\alpha = \frac{p_{\alpha_1} - p_{\alpha_2}}{K_\alpha p_{dyn}} \qquad \beta = \frac{p_{\beta_1} - p_{\beta_2}}{K_\beta p_{dyn}}$$

$$p_{dyn} = p_{tot} - p_{stat} = \frac{1}{2}\rho V^2$$

- Sensor installed in front of the object
  - When this is not possible, the tubes are installed, e.g. at wingtips
- Calibration is necessary because the sensor is not mounted at CG: scale factors K(Ma), bias, time delays

Aerodynamic angles

- The Differential-Pressure Tube on the boom significantly increases the detection of objects in stealth technology
- FADS (Flush Air Data System)
  - Multiple pressure ports placed at the nose of the aircraft to measure local pressure distribution
    - One port centrally located
    - The remaining ports are placed radially
    - Port locations are selected optimally for each aircraft
    - Angle of attack, sideslip and linear speed determined using special algorithms



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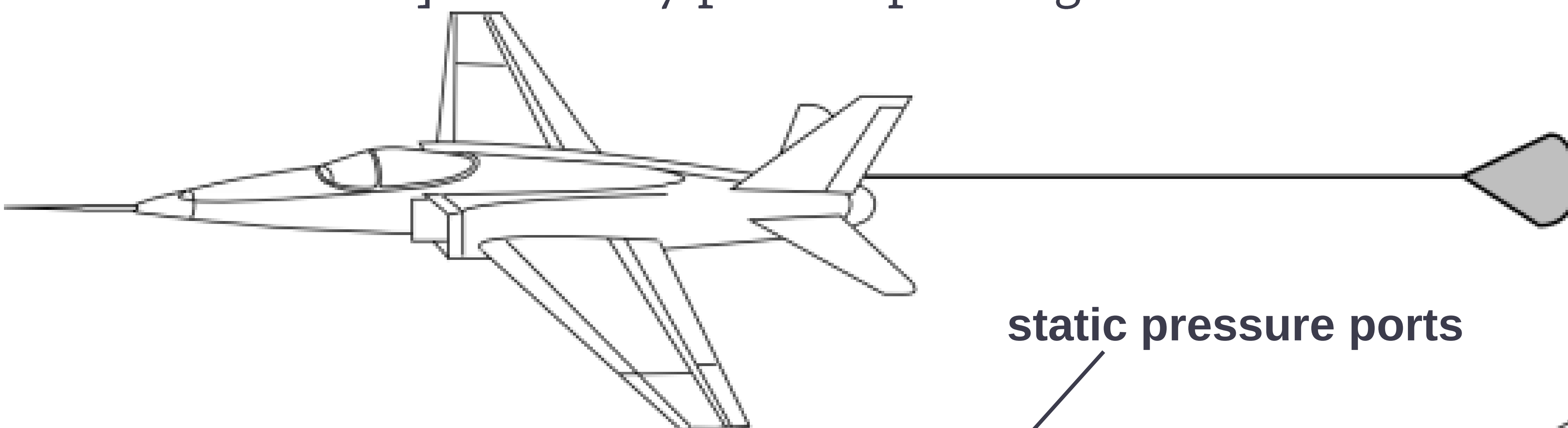
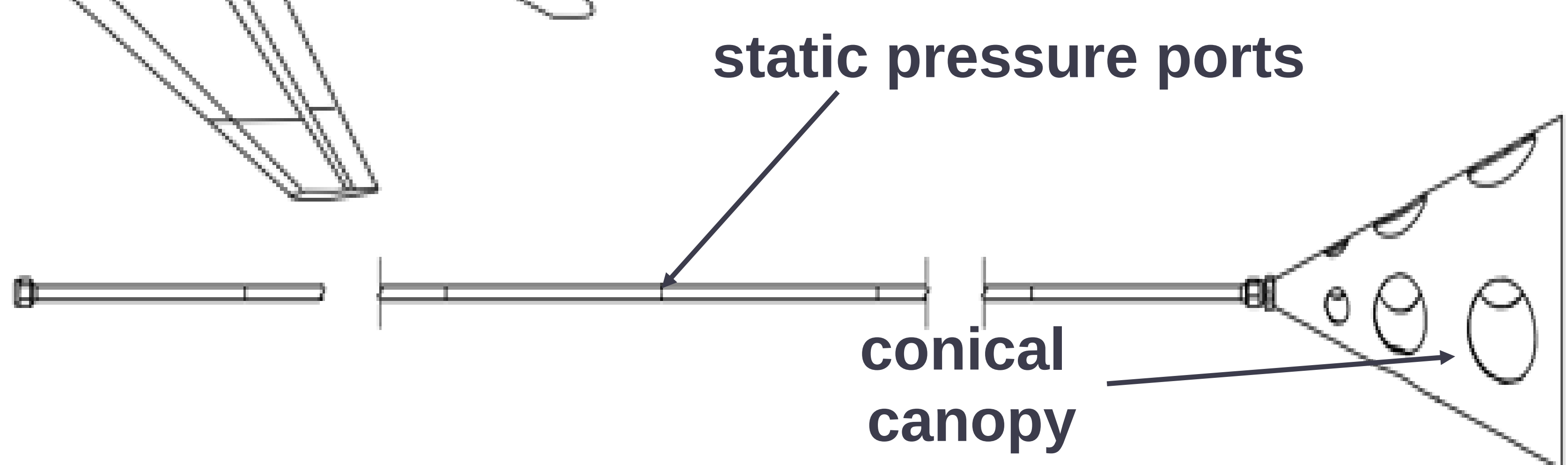
Aerodynamic angles

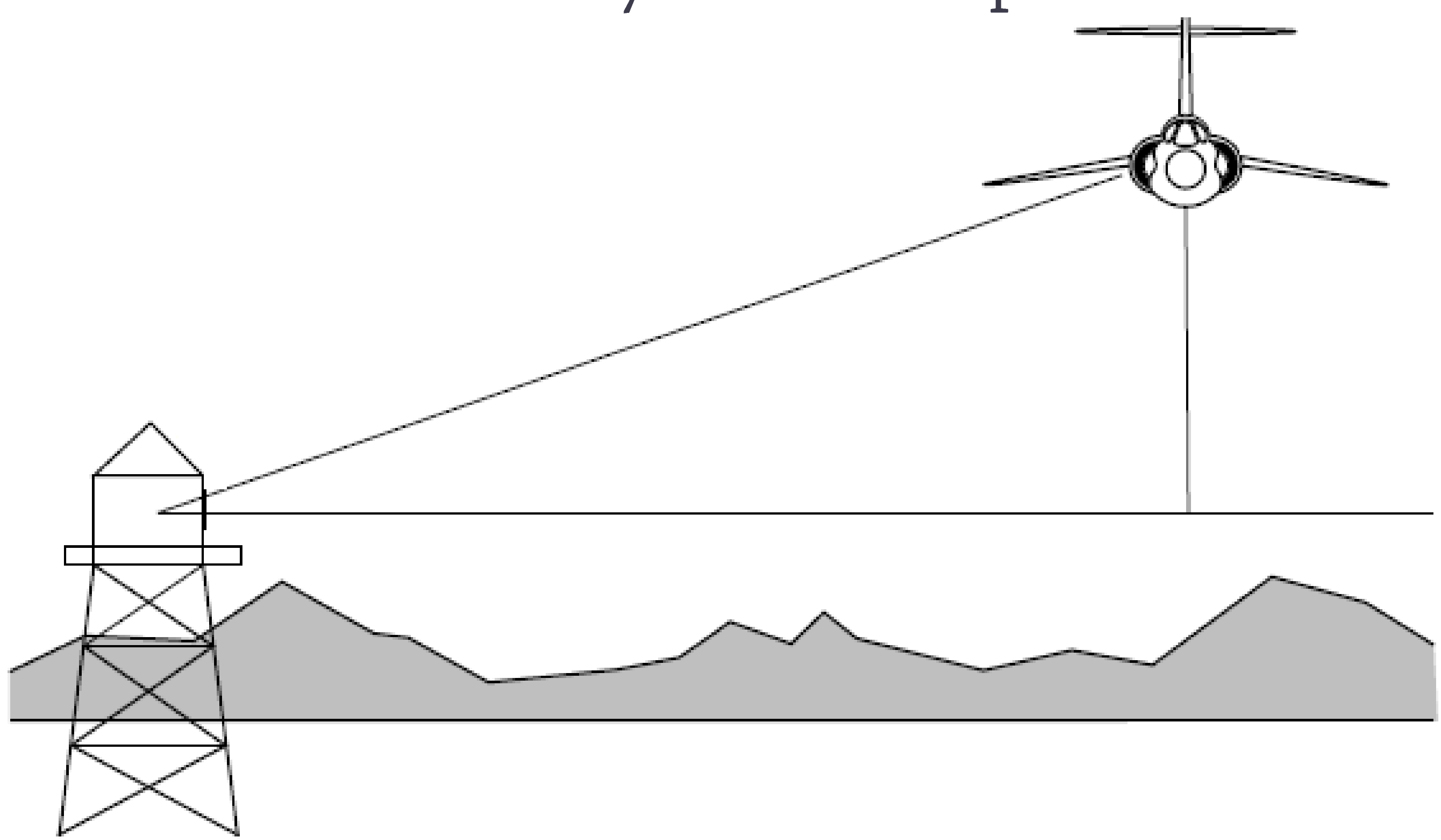
- Static pressure ports calibration
- Static pressure sensor placed behind the object in the undisturbed flow
  - Long tube equipped with static pressure ports
  - Conical, perforated canopy generating drag force to stabilize the tube
  - A mechanism for extending/retracting the sensor is required to prevent damage, e.g. during take-off (not always possible)
  - There is a risk of dynamic instabilities occurence (difficult to predict)
  - Calibration requires very precise piloting





Aerodynamic angles

- Static pressure ports calibration
  - Flight of an aircraft at a constant speed at a constant height past an observation tower
  - The height of the tower and its distance from the center of the runway are known
  - Registration of the geodetic position of an object using a camera
  - Calculation of the height of the object relative to the runway based on geometric relationships
  - Conversion of altitude above the runway into static pressure
  - Sensor calibration



Other quantities

Attitude angles

- Measured angular velocities integration (usually)
  - Attitude angles are of secondary importance

Angular accelerations

- Angular velocities differentiation
  - Rarely used in SysID
  - They contain more information about higher frequencies than other signals - can improve SysIdconvergence

Engine parameters

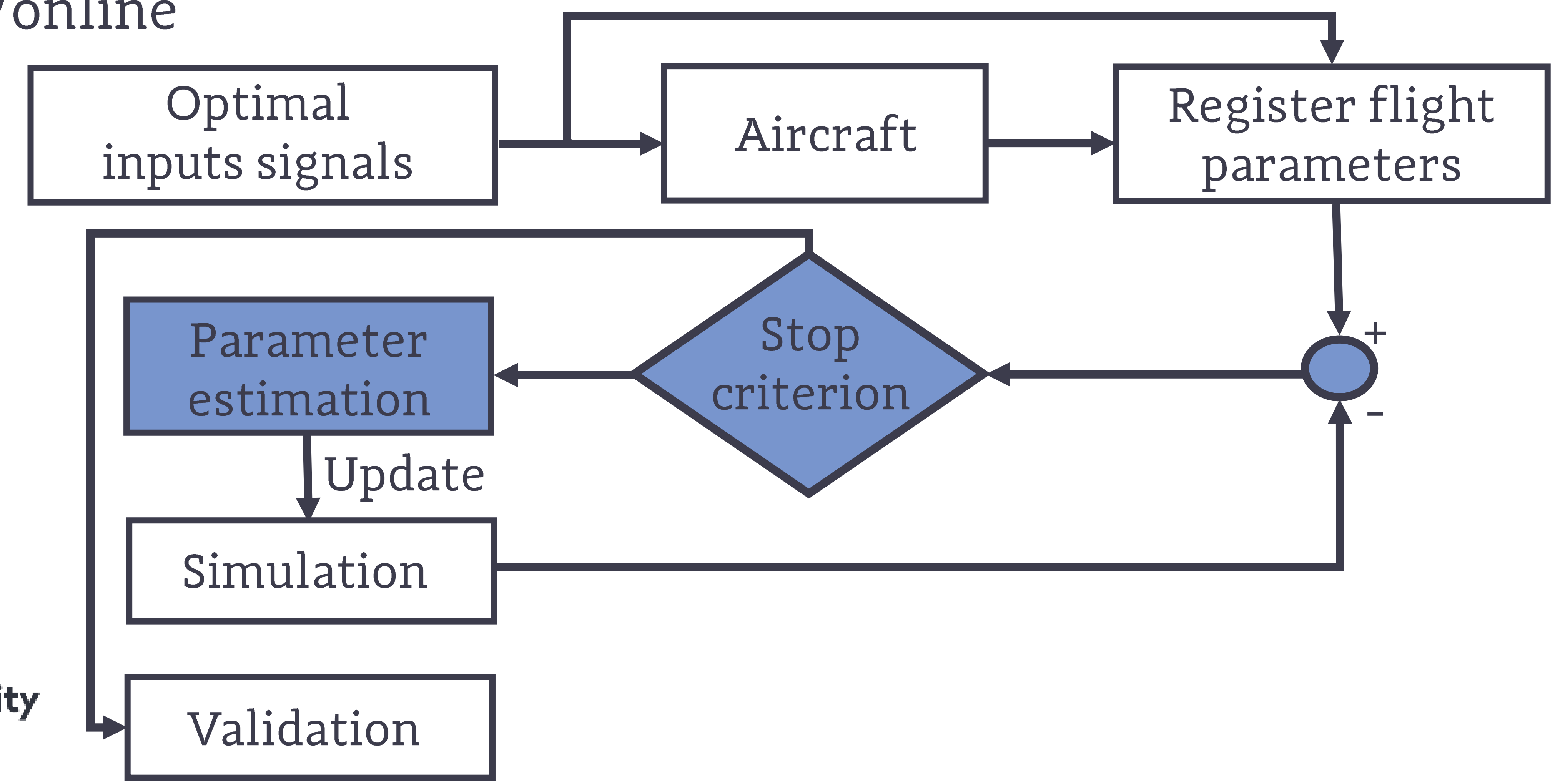
- Engine mathematical model provided by the manufacturer is used
  - Engine parameters are not typically the target of Sys-ID

Yokes

- Yokes characteristics used only for control systems modeling
- They are not used in the Sys-ID process
- In Sys-ID control surfaces deflections (not the yokes) are used as inputs.

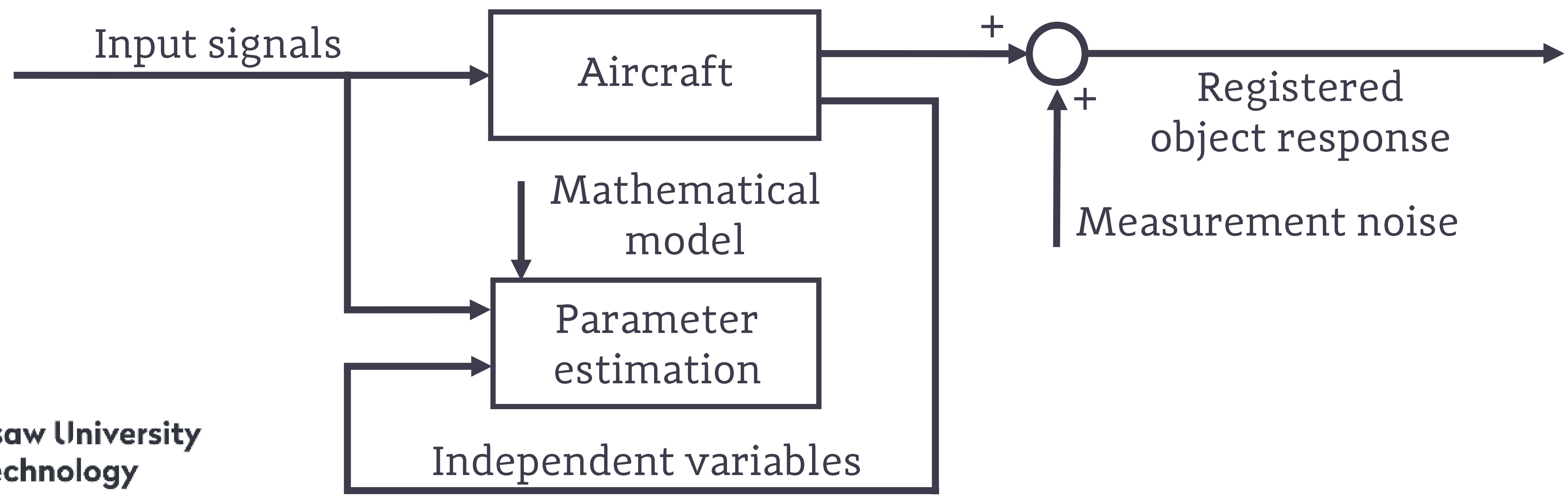
Method

- Method class
  - Equation error
  - Output Error
  - Filter error
  - Heuristics
- Domain: time/frequency
- Offline/online



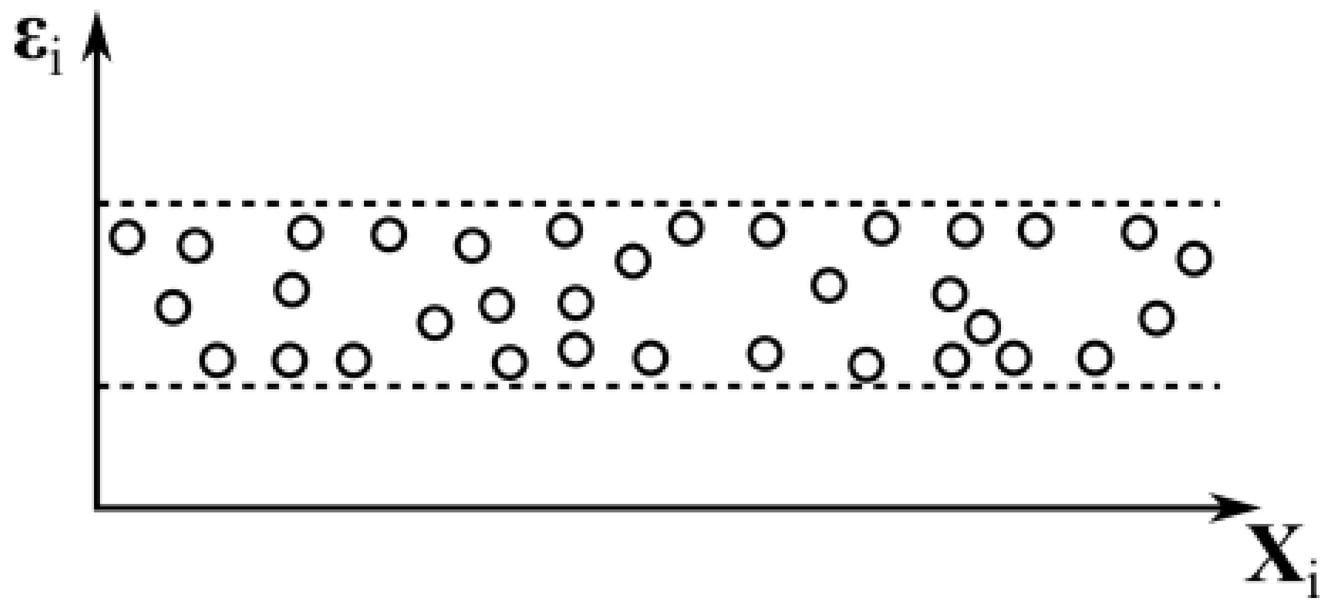
Equation error methods

- Cost function defined as a direct relationship between inputs and outputs is minimized
  - No integration is required (unstable systems)
  - Data partitioning is possible (long-lasting manoeuvres)
- All dependent and independent variables must be measured
  - Data pre-processing is required
    - Independent variables not always are directly measured
    - Systematic errors (bias, scale factors, time delays) should be removed



Ordinary Least Squares

- Oldest estimation method (Gauss, 1795)
- Ideal measurement of the independent variables **X** (not corrupted by errors or noise)
- Independent variables are not correlated
- Errors (residuals) ε of the dependent variables **Y**:
  - Uncorrelated with independent variables
  - Uniform scattered with relation to the independent variables



  - White noise with zero mean and variation σ^2
- Only linear systems can be analyzed

Ordinary Least Squares

- Linear equation describes the system

  $$\mathbf{Y} = \mathbf{X\Theta} + \boldsymbol{\varepsilon}$$

  - Sum of the squares of the resiudals is minimized to obtain the estimates

    $$\mathrm{J}(\boldsymbol{\Theta}) = \frac{1}{2}\sum_{k=1}^{n} \boldsymbol{\varepsilon}^2(\mathrm{k}) = \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\varepsilon} = \frac{1}{2}[\mathbf{Y} - \mathbf{X\Theta}]^{\mathrm{T}}[\mathbf{Y} - \mathbf{X\Theta}]$$

  - In result

    $$\widehat{\boldsymbol{\Theta}} = \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{Y}$$

- Estimation accuracy
  - Error covarinace matrix

    $$\mathbf{P} = \boldsymbol{\sigma}^2\left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}$$

    $$\widehat{\boldsymbol{\sigma}}^2 = \frac{1}{\mathrm{N} - \mathrm{n_x}}\sum_{\mathrm{k}=1}^{\mathrm{N}}\left[\mathbf{Y}(t_k) - \widehat{\mathbf{Y}}(t_k)\right]^2$$

Ordinary Least Squares

- Linear equation describes the system

  $$\mathbf{Y} = \mathbf{X\Theta} + \boldsymbol{\varepsilon}$$

  - Estimates

    $$\widehat{\mathbf{\Theta}} = \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{Y}$$

- Matrix equation is used to find the estimates
  - Unknown parameters can be find in a single step
  - A-priori values are not required
  - Integration is not required
  - Biases can be included by introducing additional independent variable as a vector with all elements equal 1
  - Equations for independent variables can be solved separately (this is a better approach because in other case some coefficients can be physical meaningless)

Ordinary Least Squares (Freq. domain)

- Discrete Fourier Transform

$$\tilde{X}(\omega) = \sum_{k=1}^{N} x(t_k) \exp(-i\omega k \Delta t)$$

  - Linear equation describes the system

    $\tilde{\mathbf{Y}} = \tilde{\mathbf{X}}\boldsymbol{\Theta} + \tilde{\boldsymbol{\varepsilon}}$

    - The cost function is analogous to that in the time domain

$$J(\boldsymbol{\Theta}) = \frac{1}{2}\left[\tilde{\mathbf{Y}} - \tilde{\mathbf{X}}\boldsymbol{\Theta}\right]^{\dagger}\left[\tilde{\mathbf{Y}} - \tilde{\mathbf{X}}\boldsymbol{\Theta}\right]$$

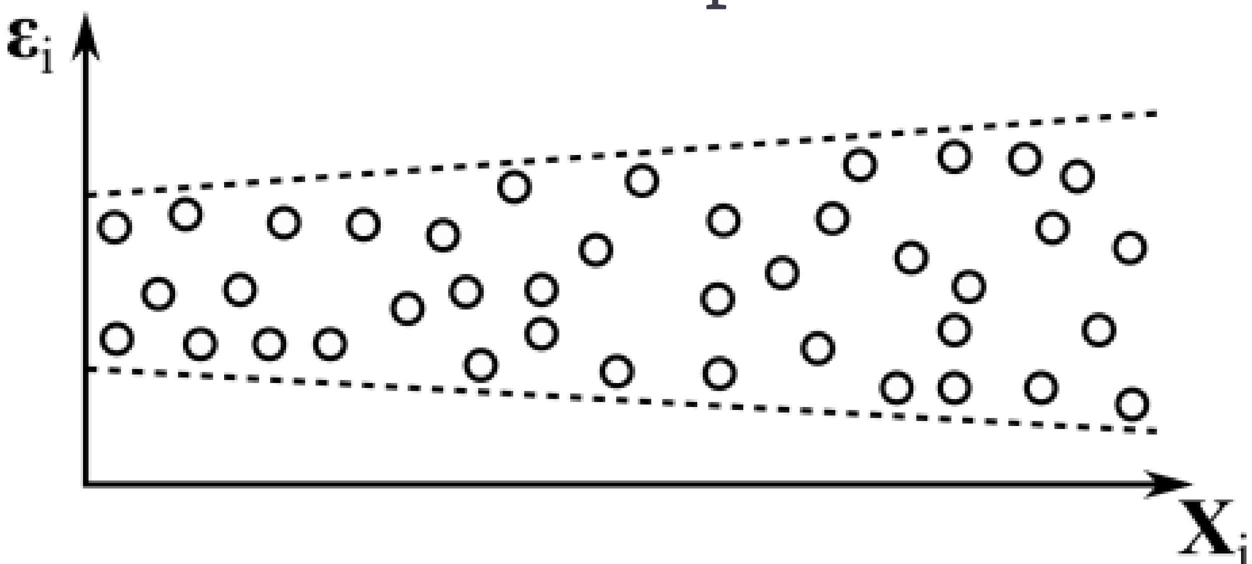
    - As a result

$$\hat{\boldsymbol{\Theta}} = \left(\mathrm{Re}\left(\tilde{\mathbf{X}}^{\dagger}\tilde{\mathbf{X}}\right)\right)^{-1} \mathrm{Re}\left(\tilde{\mathbf{X}}^{\dagger}\tilde{\mathbf{Y}}\right)$$

- Inability to identify aerodynamic derivatives biases, e.g. CL0
  - Subtracting flight parameter trim point values from their perturbed values is required
  - Analysis only in the selected frequency range - filtering noise outside this range

Weighted Least Squares

- The residuals ε dispertion for the dependent variables Y is uneven in relation to the independent variables



- Introduction of the weight coefficient matrix w_k

$$J(\boldsymbol{\Theta})\frac{1}{2}\sum_{k=1}^{n} w_k \boldsymbol{\varepsilon}^2(t_k) = \frac{1}{2}\boldsymbol{\varepsilon}^T \mathbf{W} \boldsymbol{\varepsilon} = \frac{1}{2}[\mathbf{Y} - \mathbf{X}\boldsymbol{\Theta}]^T \mathbf{W} [\mathbf{Y} - \mathbf{X}\boldsymbol{\Theta}]$$

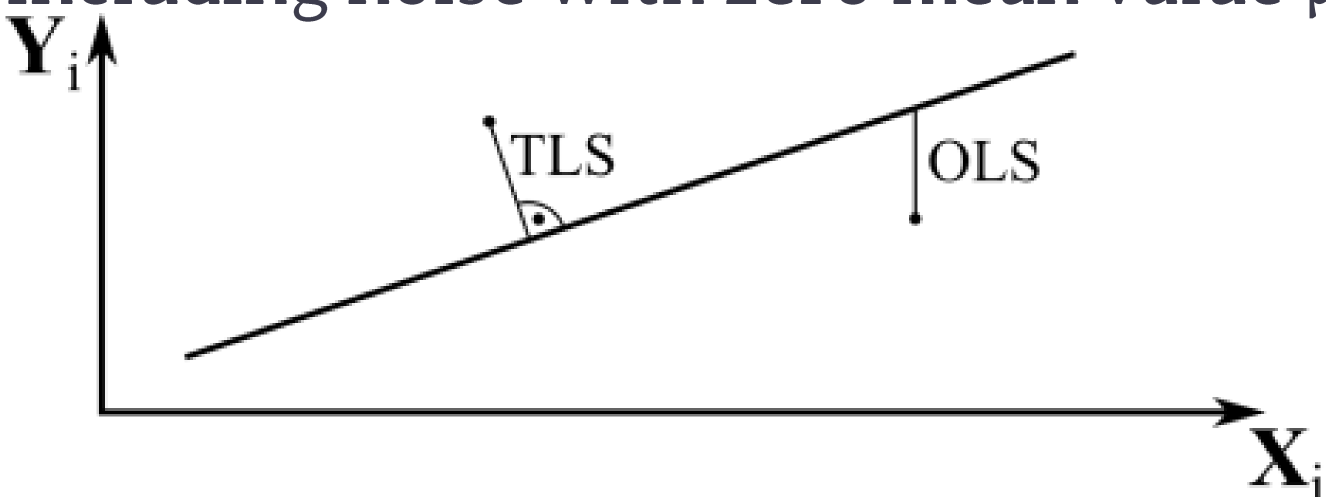
- As a result

$$\widehat{\boldsymbol{\Theta}} = \left(\mathbf{X}^T \mathbf{W} \mathbf{X}\right)^{-1} \mathbf{X}^T \mathbf{W} \mathbf{Y}$$

- Weighting factors are usually selected to be inversely proportional to the variance of the independent variables

Total Least Squares

- Including noise with zero mean value μ also in independent variables



- A linear system described by the equation

  $$\mathbf{Y} = (\mathbf{X} - \boldsymbol{\mu})\boldsymbol{\Theta} + \boldsymbol{\varepsilon}$$

  - Can be expressed as

    $$[[\mathbf{Y}|\mathbf{X}] \quad -[\boldsymbol{\mu}|\boldsymbol{\varepsilon}]]\begin{bmatrix}\boldsymbol{\Theta} \\ -\mathbf{1}\end{bmatrix} = \mathbf{0}$$

    - Estimates are obtained by singular value decomposition

      $$[\mathbf{Y}|\mathbf{X}] = [\mathbf{U}_\mathrm{S} \quad \mathbf{U}_\mathrm{N}]\begin{bmatrix}\boldsymbol{\Sigma}_\mathrm{S} & 0 \\ 0 & \boldsymbol{\Sigma}_\mathrm{N}\end{bmatrix}\begin{bmatrix}\mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22}\end{bmatrix}^\mathrm{T}$$

      $$\widehat{\boldsymbol{\Theta}} = -\mathbf{V}_{12}\mathbf{V}_{22}^{-1}$$

Instrumental variables

- Introducing instrumental variables Z to account for correlated noise in the independent variables μ
- Instrumental variables selected to be (at the same time):
  - Highly correlated with the independent variables
  - Uncorrelated with the residuals

- Methods of selecting instrumental variables:
  - Filtration based on a-priori estimates or estimates obtained from OLS
  - Time-lagged independent variables used as instrumental variables

- Model parameter estimates:
  $$\hat{\mathbf{\Theta}} = \left(\mathbf{Z}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{Z}^{\mathrm{T}}\mathbf{Y}$$

Independent variables colinearity

- Independent variables colinearity detection
  - Information matrix eigenvectors and eigenvalues

$$\mathbf{X}^{T}\mathbf{X} = \mathbf{T\Lambda T}^{-1}$$

T – eigenvectors matrix
Λ – eigenvalues λ diagonal matrix

    - Very small eigenvalues of the information matrix
    - Matrix conditioning index much greater than 1

$$C_i = \frac{|\lambda_{max}|}{|\lambda_i|}$$

  - Singular value decomposition of the independent variables matrix

$$\mathbf{X} = \mathbf{U\Sigma V}^{-1}$$

Σ – diagonal singular values σ matrix

    - Matrix conditioning index much greater than 1

$$C_i = \frac{\sigma_{max}}{\sigma_i}$$

OLS - Mixed estimation

- Directly adding a set of known a priori values of estimated parameters to the set of measurement data

$$\mathbf{\Theta}_0 = \mathbf{A}\mathbf{\Theta} + \boldsymbol{\zeta}$$

$$\mathrm{E}\{\boldsymbol{\zeta}\boldsymbol{\zeta}^{\mathrm{T}}\} = \sigma^2\mathbf{W}$$

$$\mathrm{E}\{\boldsymbol{\zeta}\} = 0$$

$\mathbf{\Theta}_0$ - a-priori values vector
$\mathbf{A}$ - marix with known constants, based on a-priori knowledge

  - New problem formulation

$$\begin{bmatrix}\mathbf{Y}\\ \mathbf{\Theta}_0\end{bmatrix} = \begin{bmatrix}\mathbf{X}\\ \mathbf{A}\end{bmatrix}\mathbf{\Theta} + \begin{bmatrix}\boldsymbol{\varepsilon}\\ \boldsymbol{\zeta}\end{bmatrix}$$

    - Model parameter estimates:

$$\hat{\mathbf{\Theta}} = \left(\mathbf{X}^{\mathrm{T}}\mathbf{X} + \mathbf{A}^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{A}\right)^{-1}\left(\mathbf{X}^{\mathrm{T}}\mathbf{Y} + \mathbf{A}^{\mathrm{T}}\mathbf{W}^{-1}\right)\mathbf{\Theta}_0$$

- Accurate knowledge of $\mathbf{\Theta}_0$ and $\mathbf{W}$ is required
- Including preliminary knowledge allows to reduce estimates variance, i.e. mixed estimation is an indirect way to solve problems with correlation between independent variables

Nonlinear least squares

- Types of nonlinearities present in the model:
  - Linear model parameters and non-linear independent variables

    $y = \Theta_1 x_1 + \Theta_2 x_2^2 + \cdots$

    - Evaluating non-linear independent variables and treating them as pseudo-signals

      $x_2^* = x_2^2$

      $y = \Theta_1 x_1 + \Theta_2 x_2^* + \cdots$
  - Nonlinearities in the model
    - Cost function

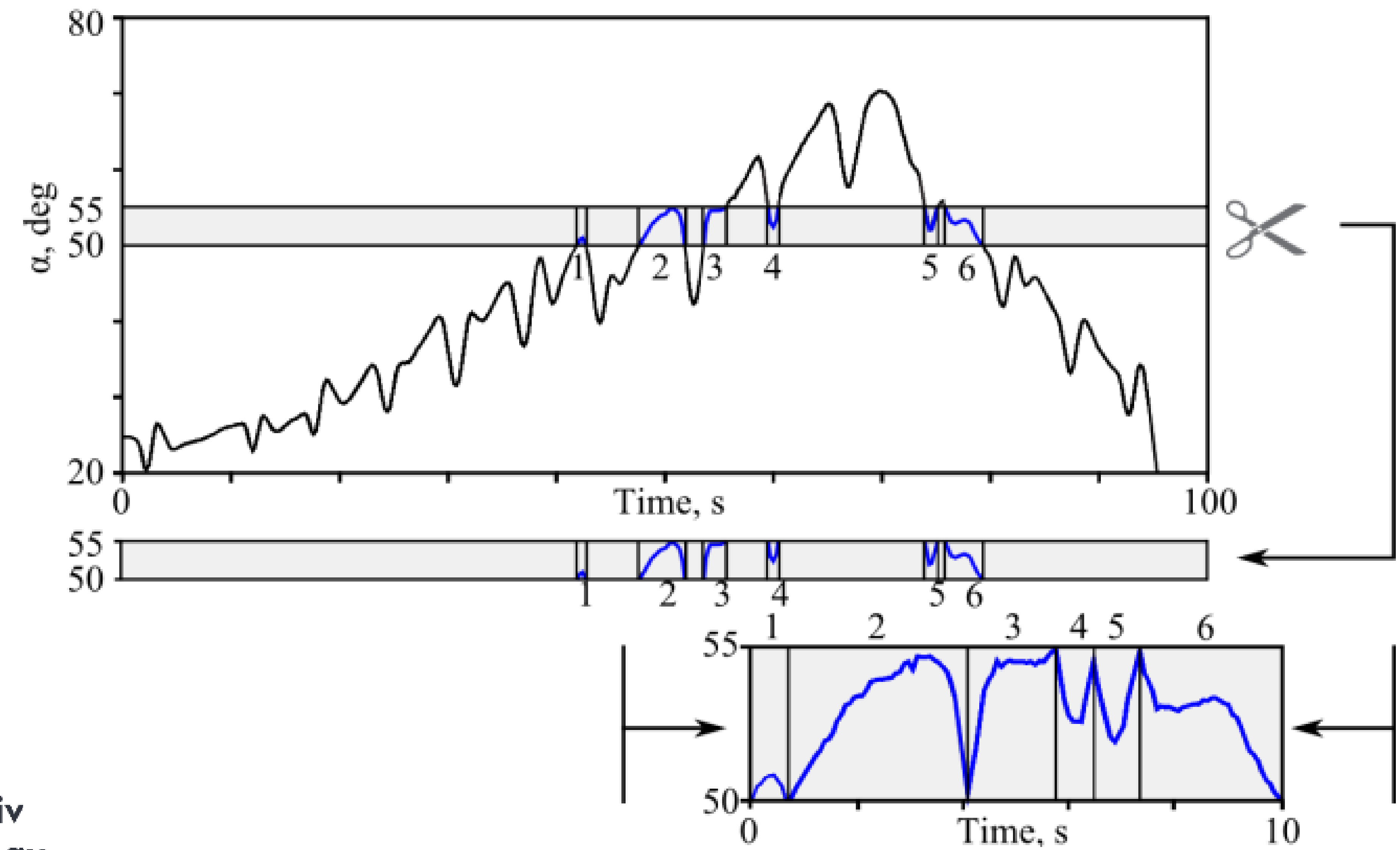
      $y = f(\mathbf{x}, \boldsymbol{\Theta})$

      - Iterative algorithm to minimize the objective function (determine estimates) is used

$$J(\boldsymbol{\Theta}) = \sum_{k=1}^{N} [y(t_k) - f(\mathbf{x}(t_k), \boldsymbol{\Theta})]^2$$

Data partitioning

- Used to analyze long-lasting maneuvers
  - Selecting of time windows in which the estimated parameters values are the same (i.e. equivalent flight conditions)
  - Combining time windows does not require bias estimation (unlike in the output error and filter error methods)



Model structure modification

- Regression analysis
  - Independent and dependent variables are uncorrelacted
  - Independent variables are uncorrelated with each other

  - Methods for determining model structure (linear regression)
    - Forward selection
      1. Assume that the model structure contains only biases
      2. Calculate correlation coefficients between independent variables and each dependent variable
      3. Include the independent variable with the highest correlation coefficient in the model structure
      4. Complete the model with additional independent variables until the t statistic is greater than the assumed threshold
    - Backwards elimination
    - Stepwise regression
      - Combines the features of forward selection and backward elimination to select the best set of independent variables

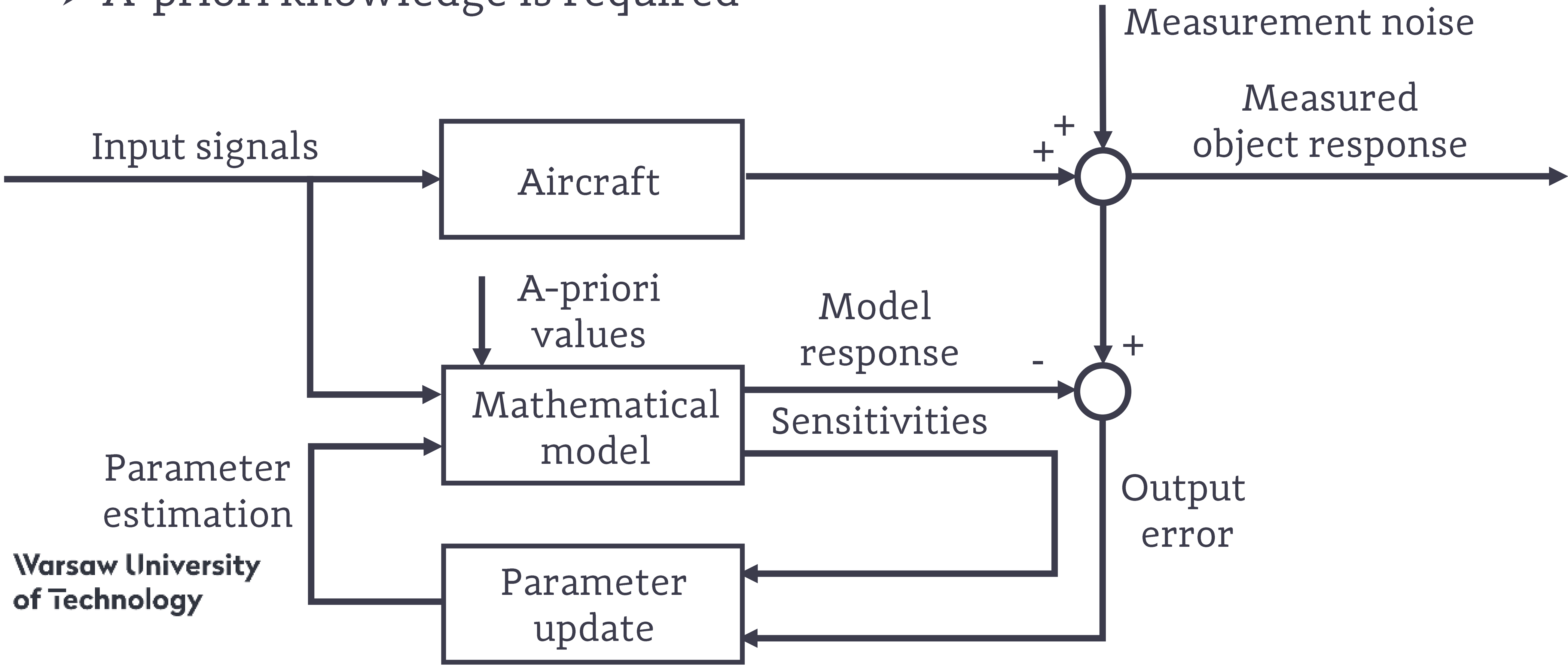
Stepwise regression

1. Calculate correlation coefficients between independent variables and each dependent variable
   - Include the independent variable with the highest correlation coefficient in the model structure
2. Calculate partial correlation for the remaining variables
   - Include the independent variable with the highest partial correlation in the model
3. Calculate the F (or t) statistic for all variables included already in the model and remove those with statistics below a specified threshold
4. Return to step 2 until none of the remaining variables lead to an improvement in the model

- The coefficient of determination R^2 can also be used for elimination
  - R^2 improves when independent variables are added to the model (even if they do not affect the object)
    - It is better to use the adjusted coefficient of determination $AdjR^2$

Output Error Method

- Minimizes the error between the model response and the measured object response
  - Most popular method
  - No process noise (e.g. turbulence, gust)
  - Difficulties with application for unstable systems (when performed in time domain) – integration problems
  - A-priori knowledge is required



Maximum Likelihood Principle

- For a deterministic system a set of parameters that maximizes the probability
  of observing measurements is searched

$$\hat{\mathbf{\Theta}} = \arg\left(\max_{\mathbf{\Theta}} p(\mathbf{z}|\mathbf{\Theta})\right)$$

- Conditional probability
  - Multivariate normal distribution
  - Output errors are independent for every time point

$$\mathrm{p}(\mathbf{z}|\mathbf{\Theta}) = \frac{1}{\sqrt{(2\pi)^{n_y}|\mathbf{R}|}^{N}} \exp\left(-\frac{1}{2}\sum_{\mathrm{k}=1}^{\mathrm{N}}[\mathbf{z}(\mathrm{t_k}) - \mathbf{y}(\mathrm{t_k})]^{\mathrm{T}}\mathbf{R}^{-1}[\mathbf{z}(\mathrm{t_k}) - \mathbf{y}(\mathrm{t_k})]\right)$$

$$\mathrm{J}(\mathbf{\Theta}) = -\ln \mathrm{L}(\mathbf{\Theta}|\mathbf{z})$$

- Cost function – negative log-likelihood

$$\mathrm{L}(\mathbf{\Theta}|\mathbf{z}) = \mathrm{p}(\mathbf{z}|\mathbf{\Theta})$$

Maximum Likelihood Principle

- Cost function minimization

$$J(\boldsymbol{\Theta}) = \frac{1}{2}\sum_{k=1}^{N}[\mathbf{z}(t_k) - \mathbf{y}(t_k)]^T\mathbf{R}^{-1}[\mathbf{z}(t_k) - \mathbf{y}(t_k)] + \frac{N}{2}\ln\det(\mathbf{R})$$

  - Unknown covariance matrix

$$\hat{\mathbf{R}}(\boldsymbol{\Theta}) = \frac{1}{N}\sum_{k=1}^{N}[\mathbf{z}(t_k) - \mathbf{y}(t_k)][\mathbf{z}(t_k) - \mathbf{y}(t_k)]^T$$

    - Cost function

$$J(\boldsymbol{\Theta}) = \det(\mathbf{R})$$

- Optimization algorithms
  - Gauss-Newton

$$\boldsymbol{\Theta}_i = \boldsymbol{\Theta}_{i-1} - \mathbf{F}_{i-1}^{-1}\mathbf{G}_{i-1}$$

  - Levenberg-Marquard

$$\boldsymbol{\Theta}_i = \boldsymbol{\Theta}_{i-1} - \left(\mathbf{F}_{i-1}^{-1} + \lambda_i\mathbf{I}\right)\mathbf{G}_{i-1}$$

  - ...

$$\mathbf{F} = \sum_{k=1}^{N}\left[\frac{\partial\mathbf{y}(t_k)}{\partial\boldsymbol{\Theta}}\right]^T\mathbf{R}^{-1}\left[\frac{\partial\mathbf{y}(t_k)}{\partial\boldsymbol{\Theta}}\right]$$

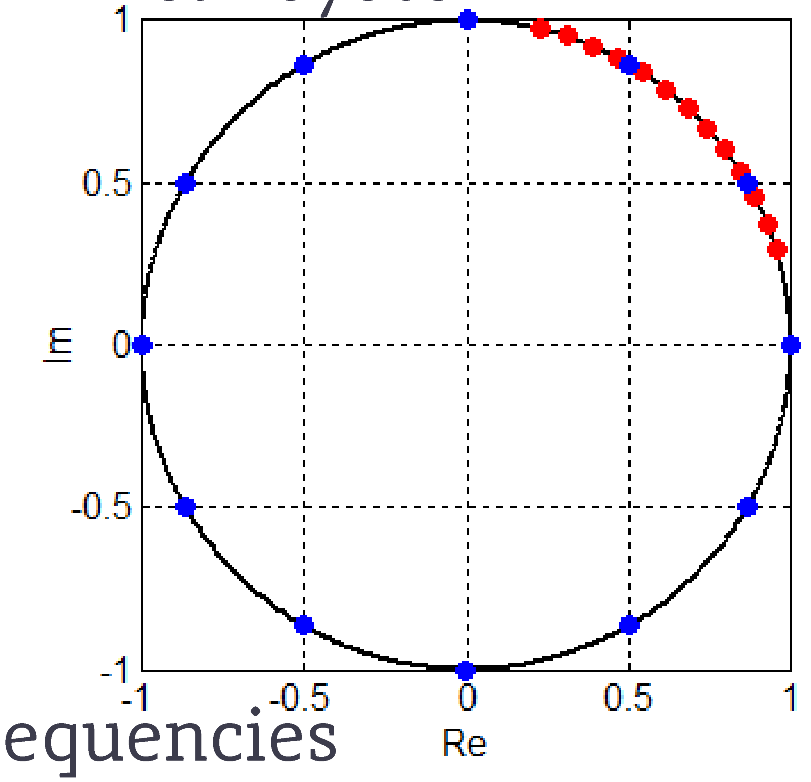
$$\mathbf{G} = -\sum_{k=1}^{N}\left[\frac{\partial\mathbf{y}(t_k)}{\partial\boldsymbol{\Theta}}\right]^T\mathbf{R}^{-1}[\mathbf{z}(t_k) - \mathbf{y}(t_k)]$$

Maximum Likelihood (Freq. domain)

- Measurements analysis in the frequency domain – linear system
  - Discrete Fourier transform

$$\tilde{X}(f_j) = \sum_{k=1}^{N} x(t_k)\, A W^{-jk}$$

$$A = 1 \quad W = \exp(i2\pi/N)$$



  - Chirp-Z transform

$$A = \exp(i\,\omega_{min}\Delta t)$$

$$W = \exp(i\,\Delta\omega\Delta t)$$

    - Possibility to define the range of analyzed frequencies
    - Ability to select frequencies for analysis

- Minimization of the objective function

$$J(\mathbf{\Theta}) = \frac{1}{2}\sum_{j=1}^{m}\left[\tilde{\mathbf{z}}(\omega_j) - \tilde{\mathbf{y}}(\omega_j)\right]^{\dagger}\mathbf{S}^{-1}\left[\tilde{\mathbf{z}}(\omega_j) - \tilde{\mathbf{y}}(\omega_j)\right] + \frac{m}{2}\ln\det(\mathbf{S})$$

  - Power spectral density of measurement error

$$\hat{\mathbf{S}}(\mathbf{\Theta}) = \frac{1}{N}\sum_{j=1}^{N}\left[\tilde{\mathbf{z}}(\omega_j) - \tilde{\mathbf{y}}(\omega_j)\right]\left[\tilde{\mathbf{z}}(\omega_j) - \tilde{\mathbf{y}}(\omega_j)\right]^{\dagger}$$

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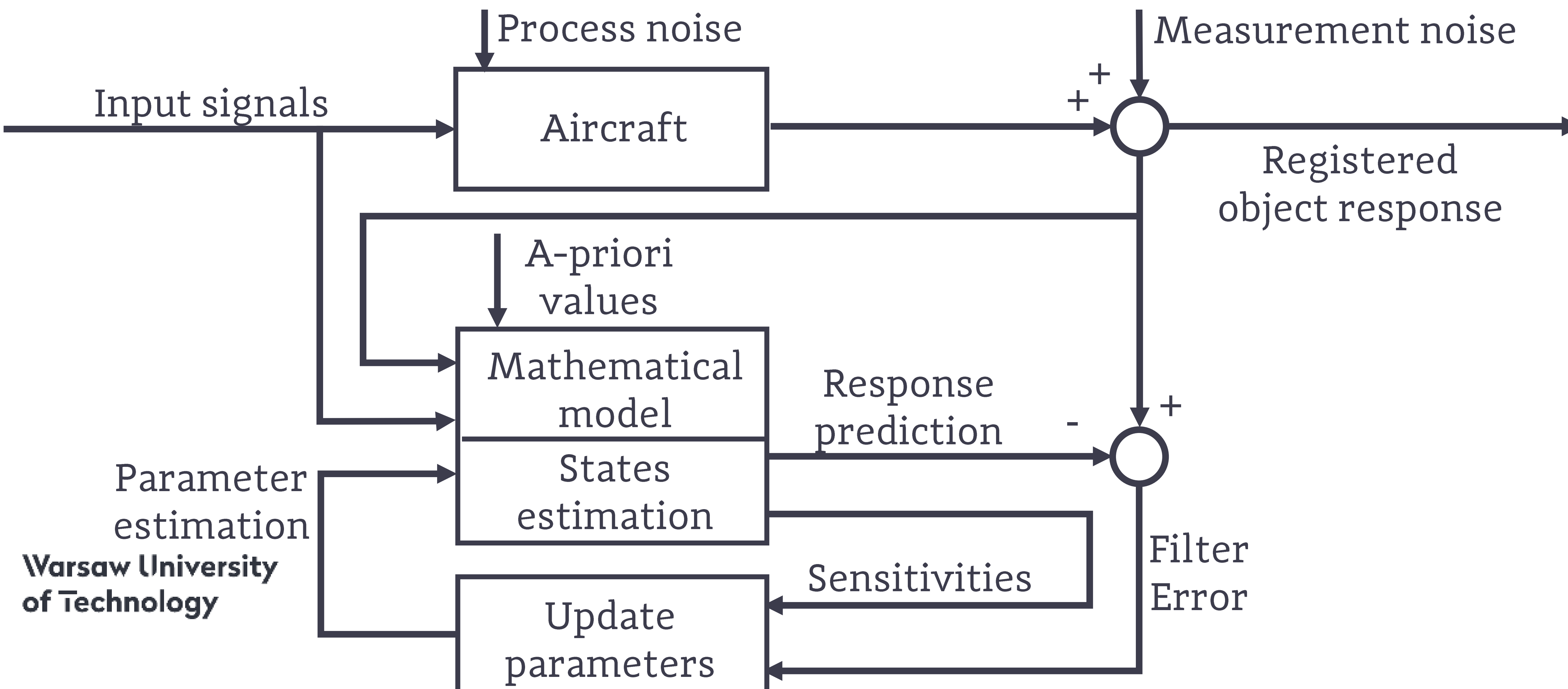
Filter Error Method

- Process and measurement noise are included
- Unstable systems can be estimated
- A-priori knowledge is required (also for process noise)
- Good match between model outputs and measurements even for wrong structure/estimates
- Problem with combining multiple manoeuvres



Filter Error Method

- Numerical integration
  - Euler
  - Runge-Kutta 4th order
  - ...

  - Discretization and application of state transition (for linear systems)

$$\Delta \mathbf{X}(t_{k+1}) = \underbrace{\mathbf{\Phi}\Delta \mathbf{X}(t_{k+1})}_{\text{Unforced component}} + \underbrace{\mathbf{\Psi}\mathbf{B}\Delta \bar{\mathbf{U}}(t_k) + \mathbf{\Psi}\mathbf{b}_x}_{\text{Forced component}}$$

- State transition matrix

$$\mathbf{\Phi} = e^{\mathbf{A}\Delta t}$$

- State transition matrix integral

$$\mathbf{\Psi} = \int_0^{\Delta t} e^{\mathbf{A}\tau} d\tau$$

  - Using Taylor's series expansion

$$\mathbf{\Phi} \approx \mathbf{I} + \mathbf{A}\Delta t + \mathbf{A}^2 \frac{\Delta t^2}{2!} + \cdots \qquad \mathbf{\Psi} \approx \mathbf{I}\Delta t + \mathbf{A}\frac{\Delta t^2}{2!} + \mathbf{A}\frac{\Delta t^3}{3!} + \cdots$$

Kalman Filter

- Linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{F}\mathbf{w}(t) + \mathbf{b}_x \quad \mathbf{x}(t_0) = 0$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{b}_y$$

$$\mathbf{z}(t_k) = \mathbf{y}(t_k) + \mathbf{G}\mathbf{v}(t_k) \quad k = 1,\dots,N$$

Distribution matrix:
F – Process noise
G – *Measurement noise*

  - Process noise v(t) and measurement noise w(t) have normal distribution with zero mean
  - Process noise and measurement noise are uncorrelated
  - Process noise and measurement noise are mutually independent

  - Prediction step

$$\tilde{\mathbf{x}}(t_{k+1}) = \mathbf{\Phi}\hat{\mathbf{x}}(t_k) + \mathbf{\Psi}\mathbf{B}\bar{\mathbf{u}}(t_k) + \mathbf{\Psi}\mathbf{b}_x$$

$$\tilde{\mathbf{y}}(t_k) = \mathbf{C}\tilde{\mathbf{x}}(t_k) + \mathbf{D}\mathbf{u}(t_k) + \mathbf{b}_y$$

$$\tilde{\mathbf{P}}(t_{k+1}) = \mathbf{\Phi}\tilde{\mathbf{P}}(t_k)\mathbf{\Phi}^T + \Delta t\mathbf{F}\mathbf{F}^T$$

  - Correction step

$$\hat{\mathbf{x}}(t_k) = \tilde{\mathbf{x}}(t_k) + \mathbf{K}(t_k)[\mathbf{z}(t_k) - \tilde{\mathbf{y}}(t_k)]$$

Kalman Filter

Covariance matrix:
R – Measurement noise
Q – Process noise

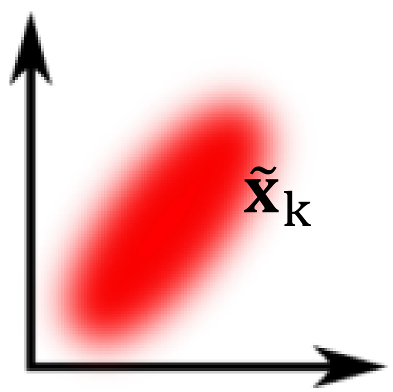
$\hat{\mathbf{x}}_0, \widehat{\mathbf{P}}_0$ - Initial values

$k = k + 1$

Prediction step

$$\widetilde{\mathbf{P}}(t_{k+1}) = \mathbf{\Phi}\widehat{\mathbf{P}}(t_k)\mathbf{\Phi}^T + \mathbf{Q}$$

$$\tilde{\mathbf{x}}(t_{k+1}) = \mathbf{\Phi}\hat{\mathbf{x}}(t_k) + \mathbf{\Psi}\mathbf{B}\mathbf{u}(t_k)$$



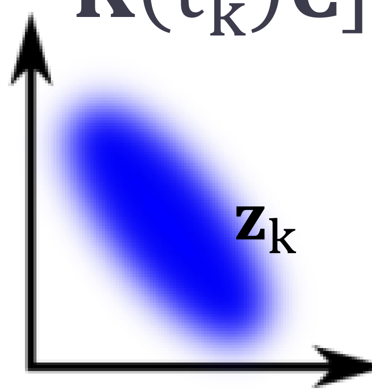
Correction step

$$\mathbf{K}(t_k) = \widetilde{\mathbf{P}}(t_k)\mathbf{C}^T\left[\mathbf{C}\widetilde{\mathbf{P}}(t_k)\mathbf{C}^T + \mathbf{R}\right]^{-1}$$

$$\tilde{\mathbf{y}}(t_k) = \mathbf{C}\tilde{\mathbf{x}}(t_k) + \mathbf{D}\mathbf{u}(t_k)$$

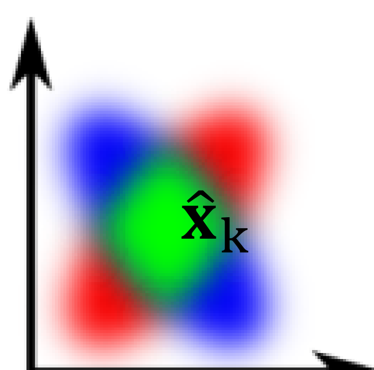
$$\hat{\mathbf{x}}(t_k) = \tilde{\mathbf{x}}(t_k) + \mathbf{K}(t_k)[\mathbf{z}(t_k) - \tilde{\mathbf{y}}(t_k)]$$

$$\widehat{\mathbf{P}}(t_k) = [\mathbf{I} - \mathbf{K}(t_k)\mathbf{C}]\widetilde{\mathbf{P}}(t_k)$$



Update

$\hat{\mathbf{x}}_k$



Kalman Filter

- Cost function

$$J(\mathbf{\Theta}) = \frac{1}{2}\sum_{k=1}^{N}[\mathbf{z}(t_k) - \tilde{\mathbf{y}}(t_k)]^T\mathbf{R}^{-1}[\mathbf{z}(t_k) - \tilde{\mathbf{y}}(t_k)] + \frac{Nn_y}{2}\ln 2\pi + \frac{N}{2}\ln\det(\mathbf{R})$$

- Unknown measurement error covariance matrix

$$\hat{\mathbf{R}}(\mathbf{\Theta}) = \frac{1}{N}\sum_{k=1}^{N}[\mathbf{z}(t_k) - \tilde{\mathbf{y}}(t_k)][\mathbf{z}(t_k) - \tilde{\mathbf{y}}(t_k)]^T$$

  - Stationary filter
    - Kalman gain matrix

$$\mathbf{K} = \mathbf{P}\mathbf{C}^T\mathbf{R}^{-1}$$

    - Error covariance matrix P is obtained by solving Riccati eq.:

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T - \frac{1}{\Delta t}\mathbf{P}\mathbf{C}^T\mathbf{R}^{-1}\mathbf{C}\mathbf{P} + \mathbf{F}\mathbf{F}^T = 0$$

- Optimization algorithm – Gauss-Newton
- Measurement noise distribution is physicaly significant for KC ≤1
  - If the condition is not met, optimize with constraints
    - Eliminate possible correlation between F and R by correcting F

Extended Kalman Filter

- Allows to identify nonlinear systems
  - Prediction step

$$\tilde{\mathbf{x}}(t_{k+1}) = \hat{\mathbf{x}}(t_k) + \int_{t_k}^{t_{k+1}} \mathbf{f}(\hat{\mathbf{x}}(t), \bar{\mathbf{u}}(t_k), \boldsymbol{\Theta})dt \quad \hat{\mathbf{x}}(t_0) = \mathbf{x}_0$$

  - Correction step

$$\tilde{\mathbf{y}}(t_k) = \mathbf{g}(\tilde{\mathbf{x}}(t_k), \mathbf{u}(t_k), \boldsymbol{\Theta}) \qquad \widetilde{\mathbf{P}}(t_{k+1}) = \boldsymbol{\Phi}\widehat{\mathbf{P}}(t_k)\boldsymbol{\Phi}^T + \Delta t \mathbf{F}\mathbf{F}^T$$

$$\hat{\mathbf{x}}(t_k) = \tilde{\mathbf{x}}(t_k) + \mathbf{K}(t_k)[\mathbf{z}(t_k) - \tilde{\mathbf{y}}(t_k)] \quad \widehat{\mathbf{P}}(t_k) = [\mathbf{I} - \mathbf{K}(t_k)\mathbf{C}]\widetilde{\mathbf{P}}(t_k)$$

- Obtaining the state and output matrix - linearization at the equilibrium point

$$\mathbf{A}_{ij} \approx \frac{\mathbf{f}_i\left(\mathbf{X}_0 + \delta\mathbf{X}_j\mathbf{e}_j, \mathbf{U}_0, \boldsymbol{\Theta}\right) - \mathbf{f}_i\left(\mathbf{X}_0 - \delta\mathbf{X}_j\mathbf{e}_j, \mathbf{U}_0, \boldsymbol{\Theta}\right)}{2\delta\mathbf{X}_j}$$

e_j – a vector containing 1 in the jth row and zeros in the rest

$$\mathbf{C}_{ij} \approx \frac{\mathbf{g}_i\left(\mathbf{X}_0 + \delta\mathbf{X}_j\mathbf{e}_j, \mathbf{U}_0, \boldsymbol{\Theta}\right) - \mathbf{g}_i\left(\mathbf{X}_0 - \delta\mathbf{X}_j\mathbf{e}_j, \mathbf{U}_0, \boldsymbol{\Theta}\right)}{2\delta\mathbf{X}_j}$$

  - Equilibrium point- updated every iteration

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Online system identification

- Acquiring instant knowledge about aircraft for adaptive control and reconfiguration
- Verification of the collected data quality and modification of planned maneuvers during the flight

- Recursive methods
  - Simplifications of more complex offline methods
  - Allow to identify systems with time-varying parameters
  - Do not require large resources in computer memory
  - Typically have slow convergence, making them unsuitable for fault detection
  - Convergence can be improved by introducing a forgetting factor, but this increases sensitivity to noise - longer data sets allow noise to be noticed
  - Continuous (uninterrupted) identification may cause numerical problems - e.g. insufficient information during steady-state flight
  - Collinearity of state and control variables is not checked

Recursive Least Squares

- Estimates updated while recording measurements
  - Based on samples collected at given time period <0; t_k>

    $$\hat{\mathbf{\Theta}}_k = \left(\mathbf{X}_k^T \mathbf{X}_k\right)^{-1} \mathbf{X}_k^T \mathbf{Y}_k \qquad \mathbf{P}_k = \left(\mathbf{X}_k^T \mathbf{X}_k\right)^{-1}$$

  - Based on samples collected at given time period <0; t_k+1>

    $$\hat{\mathbf{\Theta}}_{k+1} = \mathbf{P}_{k+1} \mathbf{X}_{k+1}^T \mathbf{Y}_{k+1}$$

    $$\mathbf{X}_{k+1} = \begin{bmatrix} \mathbf{X}_k \\ \mathbf{x}_{k+1}^T \end{bmatrix} \mathbf{Y}_{k+1} = \begin{bmatrix} \mathbf{Y}_k \\ \mathbf{y}_{k+1}^T \end{bmatrix}$$

    - Therefore

      $$\hat{\mathbf{\Theta}}_{k+1} = \mathbf{P}_{k+1}\left(\mathbf{X}_k^T \mathbf{Y}_k + \mathbf{x}_{k+1} \mathbf{y}_{k+1}\right)$$

      - Finally

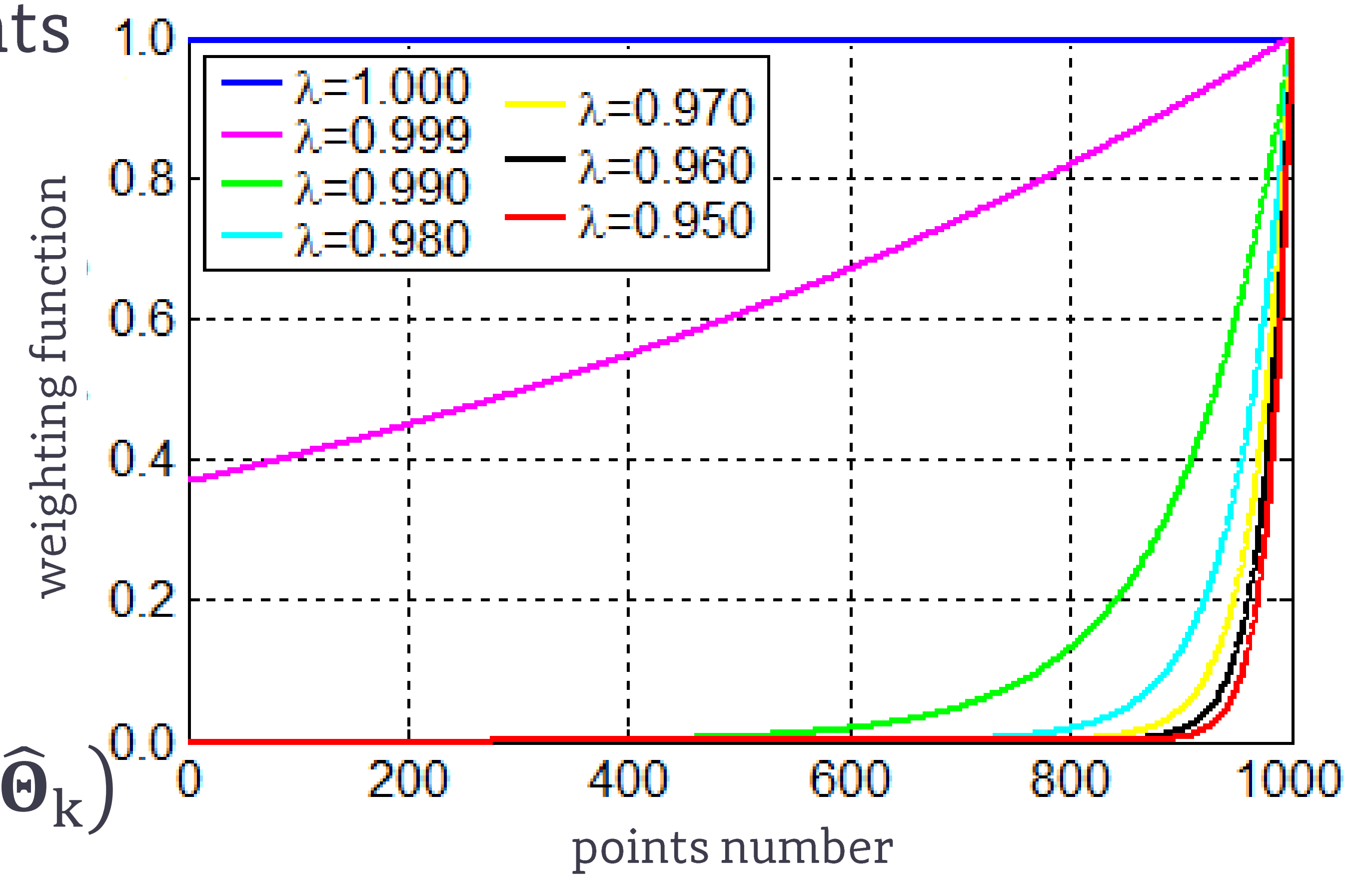
        $$\hat{\mathbf{\Theta}}_{k+1} = \hat{\mathbf{\Theta}}_k + \mathbf{K}_{k+1}\left(\mathbf{y}_{k+1} - \mathbf{x}_{k+1}^T \hat{\mathbf{\Theta}}_k\right)$$

        $$\mathbf{K}_{k+1} = \frac{\mathbf{P}_k \mathbf{x}_{k+1}}{1 + \mathbf{x}_{k+1}^T \mathbf{P}_k \mathbf{x}_{k+1}}$$

        $$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{K}_{k+1} \mathbf{x}_{k+1}^T \mathbf{P}_k$$

Recursive Least Squares

- Faster adaptation to environmental conditions
  - Introduces a weighting factor $\lambda \epsilon <0;1>$ to determe the significance of previous measurement points



- Cost function

$$J(\mathbf{\Theta}) \frac{1}{2} \sum_{i=1}^{k} \lambda^{k-i} \boldsymbol{\varepsilon}^2(i)$$

  - Updated estimate at time t_k+1

$$\widehat{\mathbf{\Theta}}_{k+1} = \widehat{\mathbf{\Theta}}_k + \mathbf{K}_{k+1}\left(\mathbf{y}_{k+1} - \mathbf{x}_{k+1}^T \widehat{\mathbf{\Theta}}_k\right)$$

$$\mathbf{K}_{k+1} = \frac{\mathbf{P}_k \mathbf{x}_{k+1}}{\lambda + \mathbf{x}_{k+1}^T \mathbf{P}_k \mathbf{x}_{k+1}}$$

$$\mathbf{P}_{k+1} = \frac{1}{\lambda}\left(\mathbf{P}_k - \mathbf{K}_{k+1} \mathbf{x}_{k+1}^T \mathbf{P}_k\right)$$

Locally Weighted Least Squares

- Removing points from the measurement data set while the set is being collected

  $$\hat{\mathbf{\Theta}} = \left(\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{Y}$$

  - Weighting matrix

    $$W_{ii} = \exp\left(-\frac{d_i}{2\kappa^2}\right)$$

  - Norm of the difference between the current point and i-th one

    $$d_i = \sum_{j=1}^{p}\left(x_{ij} - x_{hj}\right)^2$$

  - Time-varying Gaussian window width κ
    - Shorter window (smaller κ) – greater influence of newer time points

Locally Weighted Least Squares

- Removing points from the measurement data set while the set is being collected
  - Oldest
  - Containing the least information
    - Removing the measurement point (row H) after including the weights, which has the smallest contribution to the trace of the inverse of the information matrix

$$\mathbf{WX} = \begin{bmatrix} \mathbf{Z} \\ \mathbf{H} \end{bmatrix}$$

    - Matrix decomposition $P = Z^T Z + H^T H$

$$\mathbf{P} = \mathbf{USV}^T$$

    - Deleting the row with the maximum F value

$$F_i = \frac{\sum_{j=1}^{n_q} c_{ij}^2 / s_{jj}^2}{\sum_{j=1}^{n_q} c_{ij}^2 / s_{jj} - 1} \qquad \mathbf{C} = \mathbf{HV}$$

Recurisve LS (Frequency domain)

- Estimats updated during measurement registration
  - Based on samples collected at given time period <0; t_k>

$$\tilde{X}_k(\omega) = \sum_{k=1}^{N} x(t_k)\exp(-i\omega k\Delta t)$$

  - Based on samples collected at given time period <0; t_{k+1}>

$$\tilde{X}_{k+1}(\omega) = \sum_{k=1}^{N+1} x(t_{k+1})\exp(-i\omega(N+1)\Delta t) =$$

$$= \tilde{X}_k(\omega) + x(t_{k+1})\exp(-i\omega(N+1)\Delta t)$$

    - Model parameter estimates:

$$\widehat{\boldsymbol{\Theta}}_{k+1} = \left(\mathrm{Re}\left(\tilde{\mathbf{X}}_{k+1}^{\dagger}\tilde{\mathbf{X}}_{k+1}\right)\right)^{-1}\mathrm{Re}\left(\tilde{\mathbf{X}}_{k+1}^{\dagger}\tilde{\mathbf{Y}}_{k+1}\right)$$

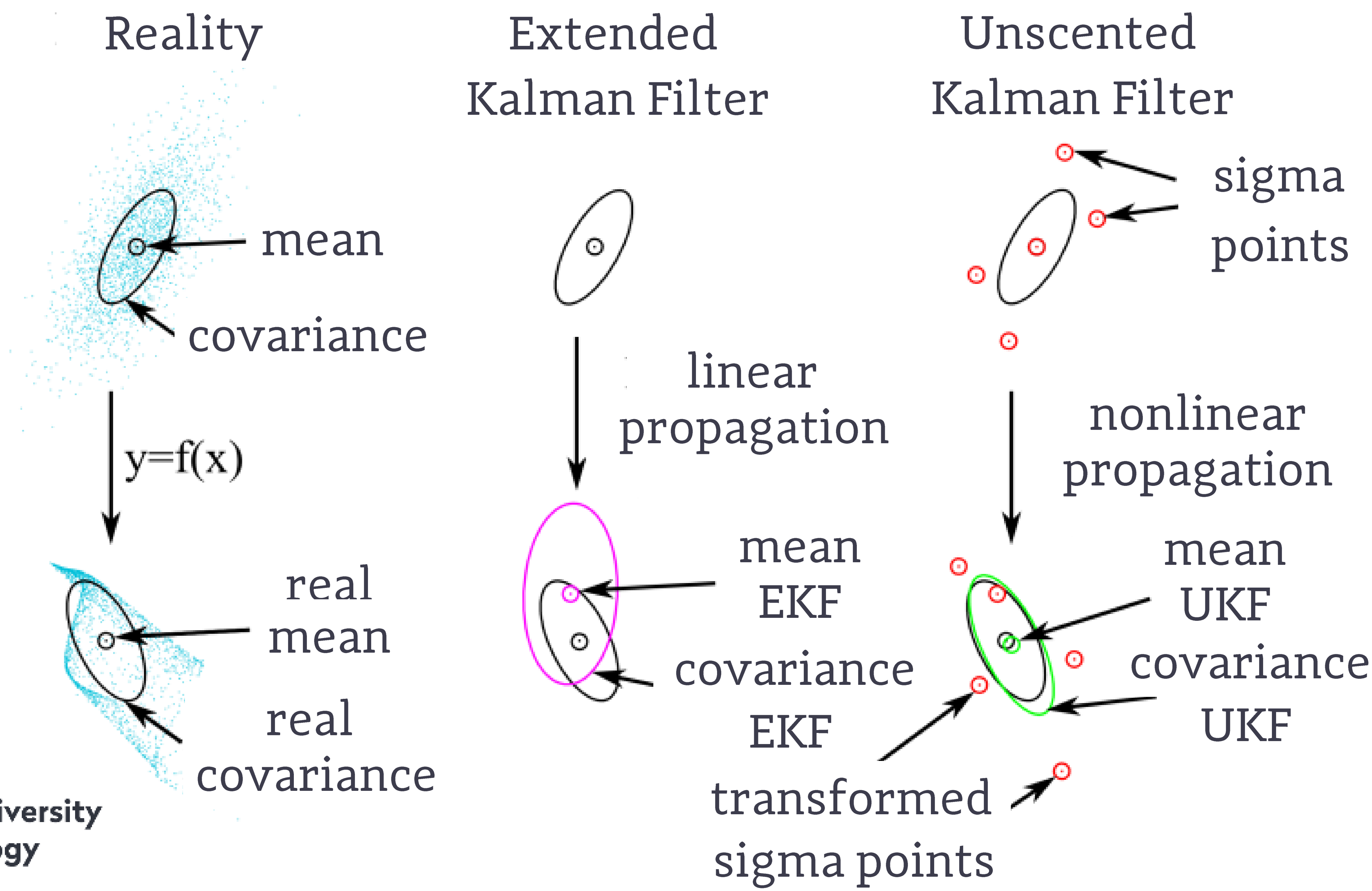
- Introducing the possibility of placing greater emphasis on current measurement data requires the use of a weighting factor or weighting matrix

Extended Kalman Filter

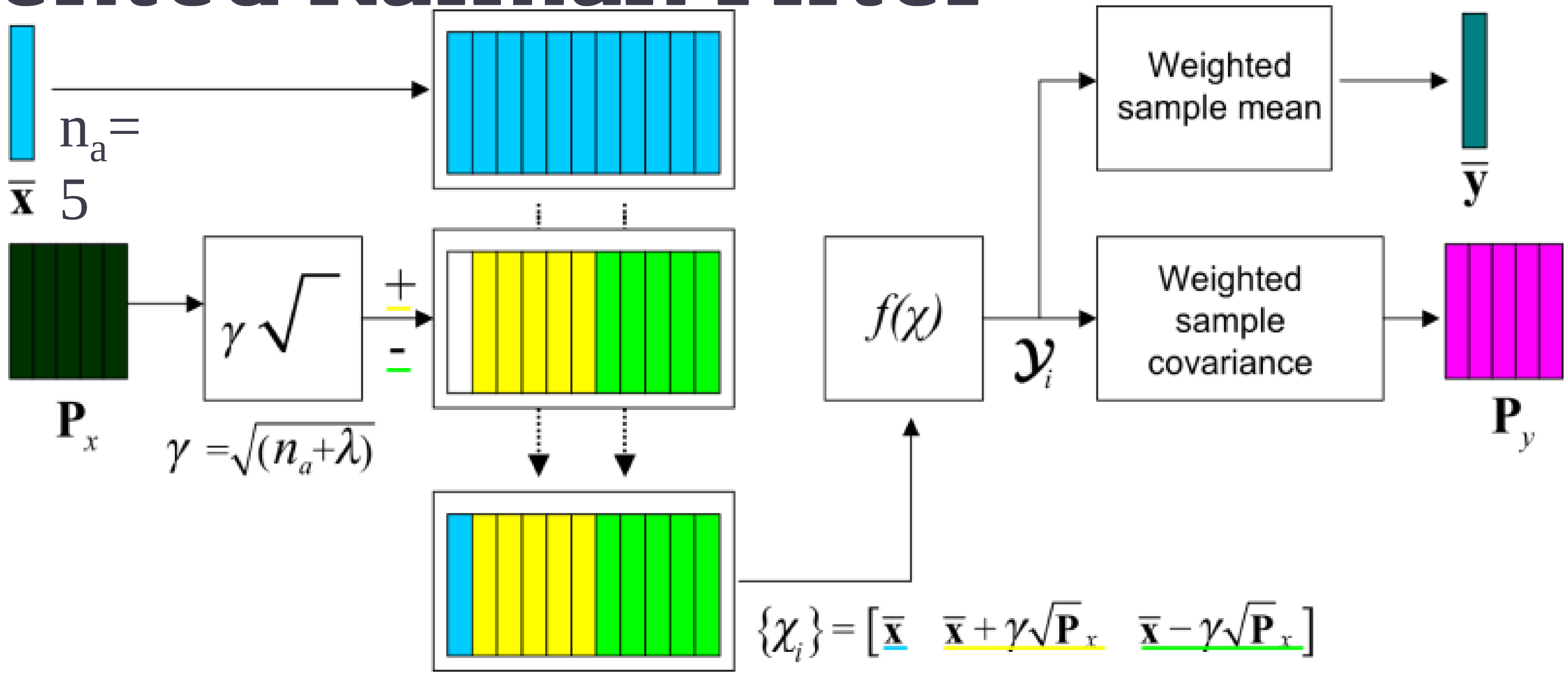
- Reformulate the problem of estimating model parameters as the problem of estimating state variables
  $\mathbf{x}_a = [\mathbf{x}^T \quad \mathbf{\Theta}^T]^T$
  - Therefore
    $$\dot{\mathbf{x}}_a = \mathbf{f}\big(\mathbf{x}_a(t), \mathbf{u}(t)\big) + \mathbf{F}_a \mathbf{w}_a(t)$$
    $$\mathbf{y}_a = \mathbf{g}\big(\mathbf{x}_a(t), \mathbf{u}(t)\big)$$
    $$\mathbf{z}(t_k) = \mathbf{y}(t_k) + \mathbf{G}\mathbf{v}(t_k) \quad k = 1,\dots,N$$
  - The prediction and correction steps are given in the same way as in the extended Kalman filter. However, due to the change in the estimated values
    $$\mathbf{\Phi}_a(t_{k+1}) \approx \mathbf{I} + \mathbf{A}_a(t_{k+1})\Delta t + \mathbf{A}_a^2(t_{k+1})\frac{\Delta t^2}{2!} + \cdots$$
    $$\widetilde{\mathbf{P}}(t_{k+1}) = \mathbf{\Phi}_a(t_{k+1})\widehat{\mathbf{P}}_a(t_k)\mathbf{\Phi}_a(t_{k+1})^T + \Delta t \mathbf{F}_a \mathbf{F}_a^T$$
    $$\mathbf{A}_a(t_k) = \left.\frac{\partial \mathbf{f}_a\big(\mathbf{x}_a(t), \mathbf{u}(t)\big)}{\partial \mathbf{x}_a}\right|_{\mathbf{x}_a = \hat{\mathbf{x}}_a(t_k)} \qquad \mathbf{C}_a(t_k) = \left.\frac{\partial \mathbf{g}_a\big(\mathbf{x}_a(t), \mathbf{u}(t)\big)}{\partial \mathbf{x}_a}\right|_{\mathbf{x}_a = \tilde{\mathbf{x}}_a(t_k)}$$

Unscented Kalman Filter

- Highly nonlinear systems
- Uses sigma points to transform mean and covariance (expressed as mean and covariance samples)



Unscented Kalman Filter

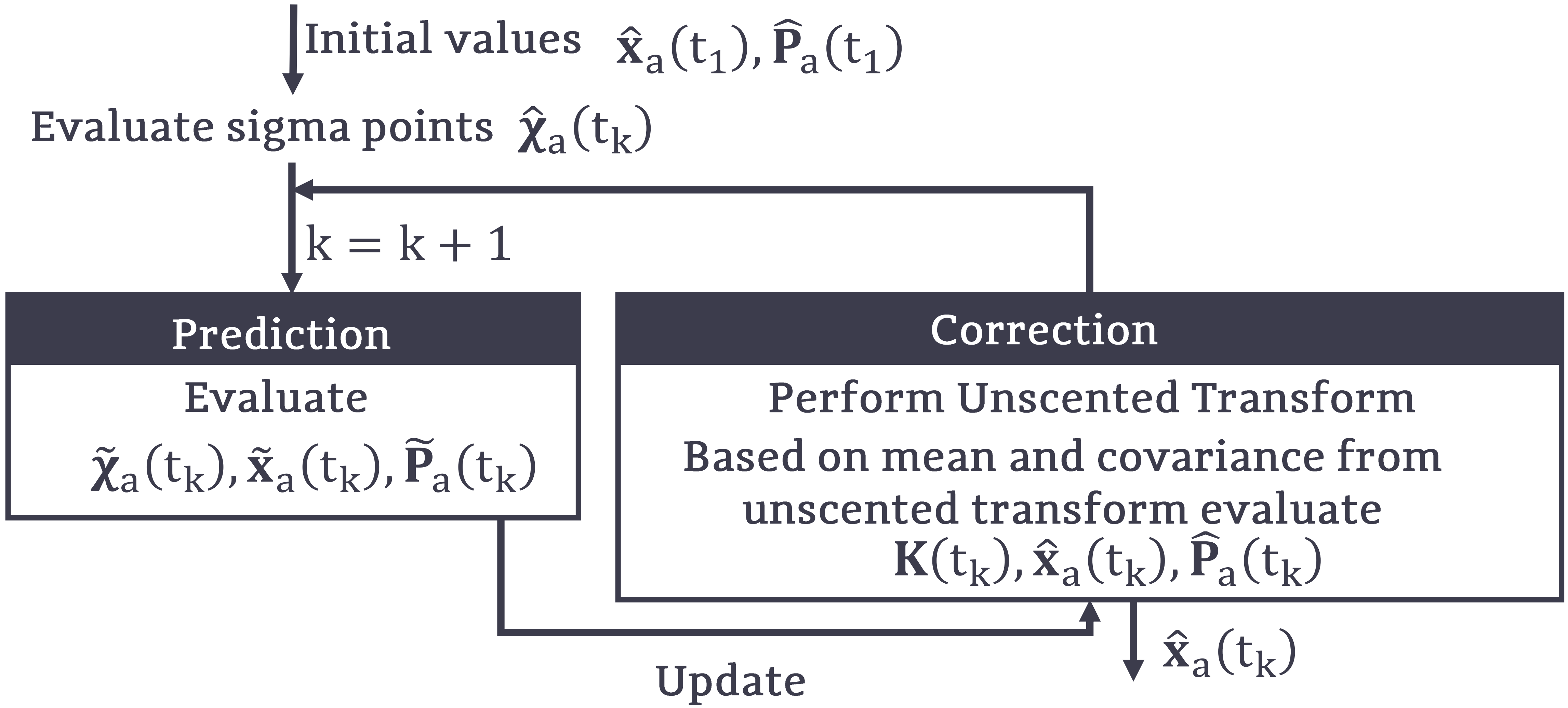


- Choice $2n_a+1$ points. sigma, n_a – number of state vector elements
- Calculate for each element of the state vector $\gamma\sqrt{\mathbf{P}_x}\ ,-\gamma\sqrt{\mathbf{P}_x}$
- Create for a set each element of the state vector:

$$\chi_i = \left[\bar{\mathbf{x}} \quad \bar{\mathbf{x}} + \gamma\sqrt{\mathbf{P}_x} \quad \bar{\mathbf{x}} - \gamma\sqrt{\mathbf{P}_x}\right]$$

- Calculate a nonlinear function $f(\chi)$ for each set
- Calculate a weighted sample for mean and covariance

Unscented Kalman Filter



- Unscented Kalman filter needs to be specify
  - Scaling factor

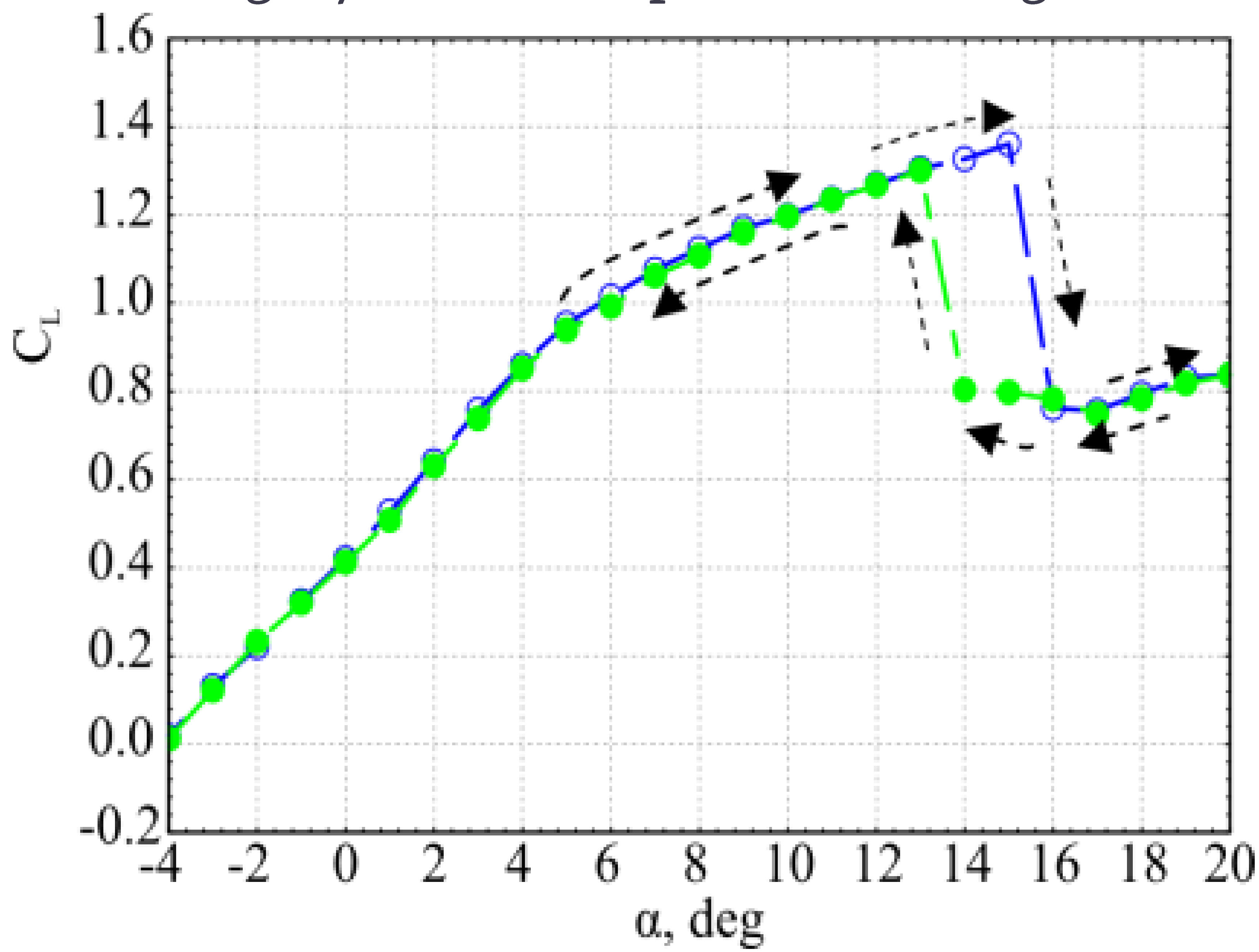
    $\lambda = \alpha^2(n_a + \kappa) - n_a$
  - Weight matrix for mean W^m and covariance W^c

    $W_0^m = \lambda/(n_a + \lambda) \quad W_0^m = \lambda/(n_a + \lambda) + (1 - \alpha^2 + \beta)$

    $W_i^m = W_i^c = 1/\big(2(n_a + \lambda)\big) \quad i = 1, \cdots 2n_a$

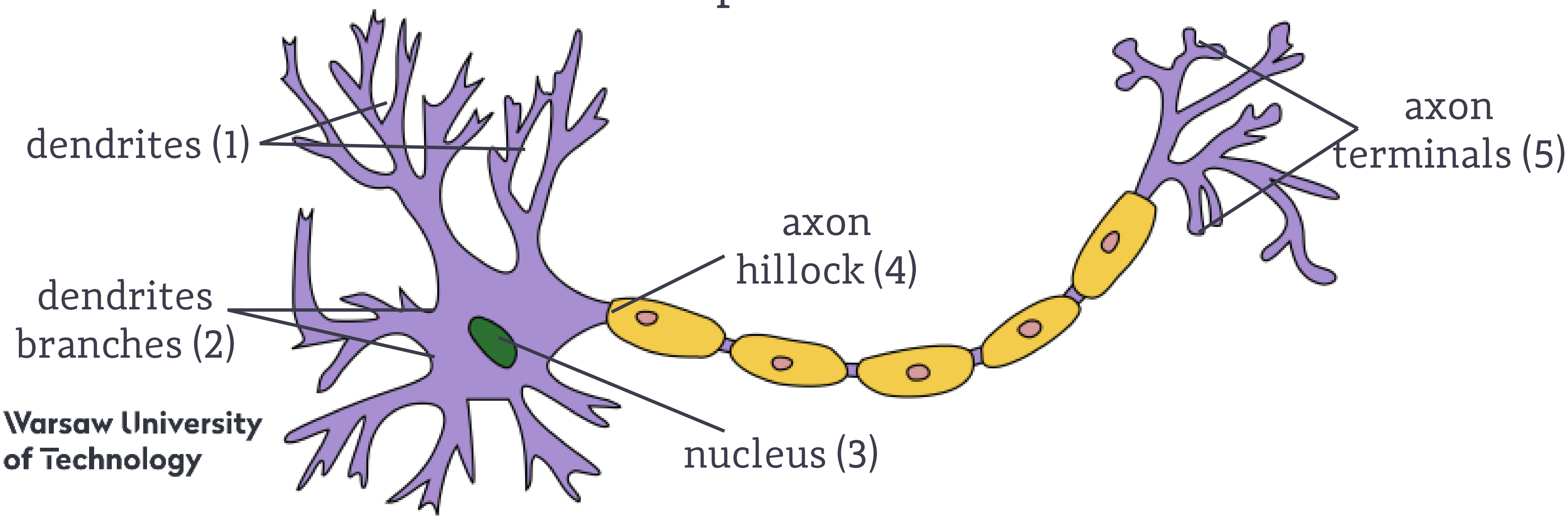
Artificial neural networks

- Mathematical structures inspired by the way in which human brain works
- Only input signals are required to obtain response of the analyzed object i.e. no model structure is assumed (behavioral models)
  - Allow to identify complex and highly nonlinear phenomena e.g. icing, lift coefficient stall hysteresis
  - Obtaining model parameters requires additional effort
- Allow to obtain model response immediately but their learning is a long-term task
- No integration is required
- A learning set must be provided



Artificial neural networks

- Neuron – electrically excitable cell capable of collecting, processing and transmitting electrical signal if the input signals combination is above a specified threshold
  - Dendrites – neuron inputs
  - Dendrite branches – ends of dendrites, place in which signals are amplified or reduced
  - Nucleus – place in which neuron essential processes are done (amplified/reduced signals are summed)
  - Axon hillock – place in which output signal is processed
  - Axon terminals – neuron outputs



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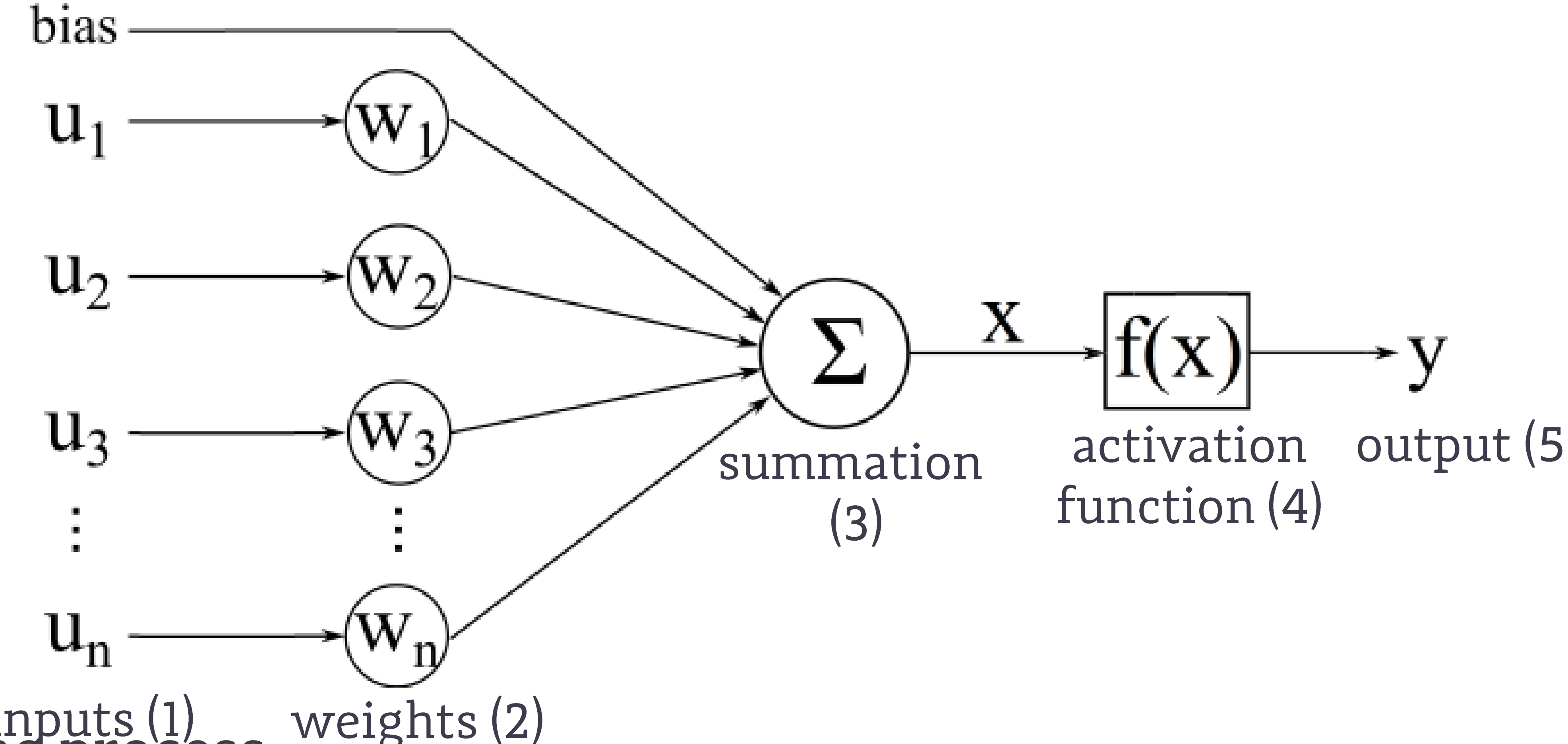
Artificial neural networks

- Neuron in artificial neural network

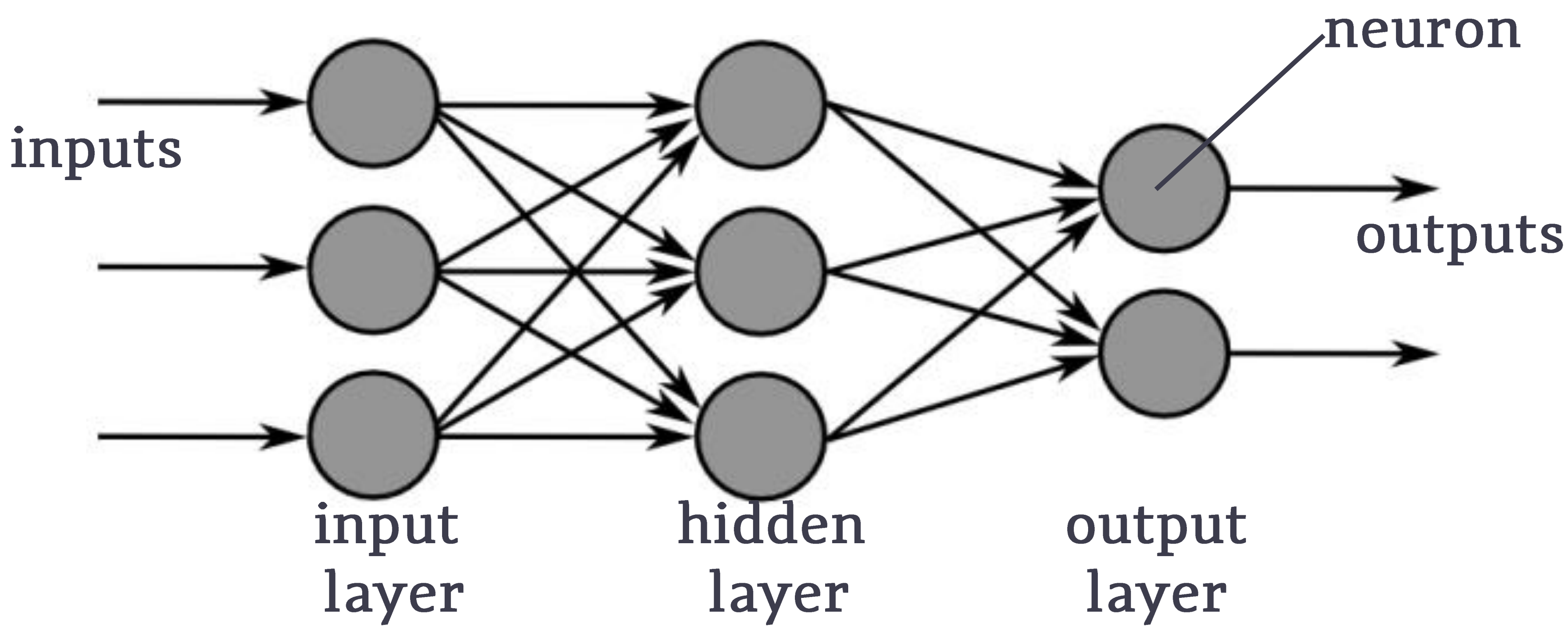


- Wining process
- Activation function:
  - Unit step
  - Linear function
  - Nonlinear function e.g. hyperbolic tangent $f_i(x_i) = \tanh\left(\frac{\gamma_i}{2} x_i\right)$

γ - slope

Artificial neural networks

- Unidirectional – one-way signal flow from input to output



- Recursive – there is feedback between input and output
- Radial – neurons in the hidden layer implement a function ϕ that changes radially around a selected center c and assumes non-zero values only in the vicinity of this center

$$\varphi(\mathbf{u},\mathbf{c}) = \varphi(\|\mathbf{u}-\mathbf{c}\|) \qquad \mathrm{y_i}(\mathbf{u}) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{w_i}\varphi(\|\mathbf{u}-\mathbf{c}_\mathrm{i}\|)$$

- …

Artificial neural networks

- Training – to determine the weights
  - Backpropagation method (with recursive change of weights)
    - Weights selected to minimize the squared output error

$$E(t_k) = \frac{1}{2}[z(t_k) - u(t_k)]^T[z(t_k) - u(t_k)]$$

- The fastest descent method

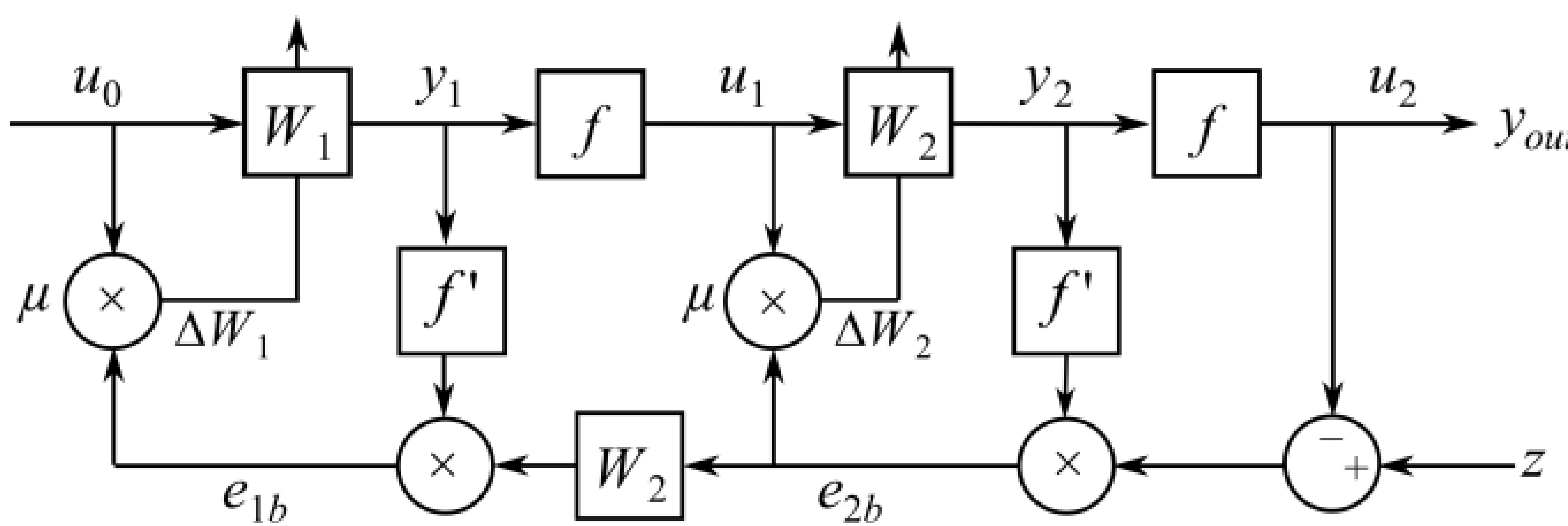
$$W_2(t_{k+1}) = W_2(t_k) + \mu\left(-\frac{\partial E}{\partial W_2}\right)_{t_k}$$

μ - learning rate

$$W_2(t_{k+1}) = W_2(t_k) + \mu e_{2b}(t_k)u_1^T(t_k)$$

$$e_{2b}(t_k) = f'[y(t_k)][z(t_k) - u_2(t_k)]$$

$$W_1(t_{k+1}) = W_1(t_k) + \mu e_{1b}(t_k)u_0^T(t_k)$$



Artificial neural networks

- Backpropagation method with moment of inertia Ω
  - Small values of learning rate result in very slow convergence
  - High values of learning rate may cause sudden jumps between weight values (especially when the cost function has numerous local extremes)

- Solution – introduce an additional coefficient causing the step to be performed in the averaged direction based on the weight values in previous iterations

$$W_1(t_{k+1}) = W_1(t_k) + \mu e_{1b}(t_k) u_0^T(t_k) + \Omega\left[W_1(t_k) - W_1\left(t_{k-1}\right)\right]$$

$$W_2(t_{k+1}) = W_2(t_k) + \mu e_{2b}(t_k) u_1^T(t_k) + \Omega\left[W_2(t_k) - W_2\left(t_{k-1}\right)\right]$$

  - The introduction of the moment of inertia coefficient $\Omega \in (0;1)$ increases the learning speed from μ to approximately $\mu/(1-\Omega)$ without causing sudden jumps in weight values

Modified backpropagation method

- Faster convergence when compared to the classic backpropagation method or the backpropagation method with moment of inertia
- Less sensitive to the settings of the artificial neural network, e.g. the activation function γ slope coefficient, weight coefficients initial values
- It introduces forgetting factors λ_1, λ_2 to increase the relevance of newer data
- Minimizes the mean squared error for the outputs from the summing block
  - Introducing a Kalman filter in each layer to determine the weighting factors

$$W_1(t_{k+1}) = W_1(t_k) + \mu e_{1b}(t_k) K_1^T(t_k)$$

$$W_2(t_{k+1}) = W_2(t_k) + [d(t_k) - y_2(t_k)] K_2^T(t_k)$$

Modified backpropagation method

- Kalman gain

$$K_1(t_k) = \frac{D_1(t_k)u_0(t_k)}{\lambda_1 + u_0^T(t_k)D_1(t_k)u_0(t_k)}$$

$$K_2(t_k) = \frac{D_2(t_k)u_1(t_k)}{\lambda_2 + u_1^T(t_k)D_2(t_k)u_1(t_k)}$$

- The inverse of the correlation matrix for network training dataset

$$D_1(t_k) = \frac{D_1(t_{k-1}) - K_1(t_{k-1})u_0^T(t_{k-1})D_1(t_{k-1})}{\lambda_1}$$

$$D_2(t_k) = \frac{D_2(t_{k-1}) - K_2(t_{k-1})u_1^T(t_{k-1})D_2(t_{k-1})}{\lambda_2}$$

- Output of the summation block

$$d(t_k) = \frac{1}{\gamma}\ln\left(\frac{1 + z(t_k)}{1 - z(t_k)}\right)$$

Artificial neural networks

- Typical neural network settings for Sys-ID

Network parameter	Value
Hidden layers number	1
Neurons in hidden layer number	5-8
Range for data scaling	from -0.5 to 0.5
Activation function slope	<0.6; 1>
Learning rate	<0.1; 0.3>
Moment of inertia	<0.3; 0.5>
Weights' initial values	<0.0; 1.0>

* Data scaling - all input signals have the same impact on the final result, leading to improved convergence

- Model parameters determination
  - Directly impossible - weights have no physical significance
  - Perturbe the selected input signal (e.g. β) twice and observe the output signal (e.g. C_l)

$$C_{l\beta} = \frac{C_l(\beta + \Delta\beta) - C_l(\beta - \Delta\beta)}{2\Delta\beta}$$

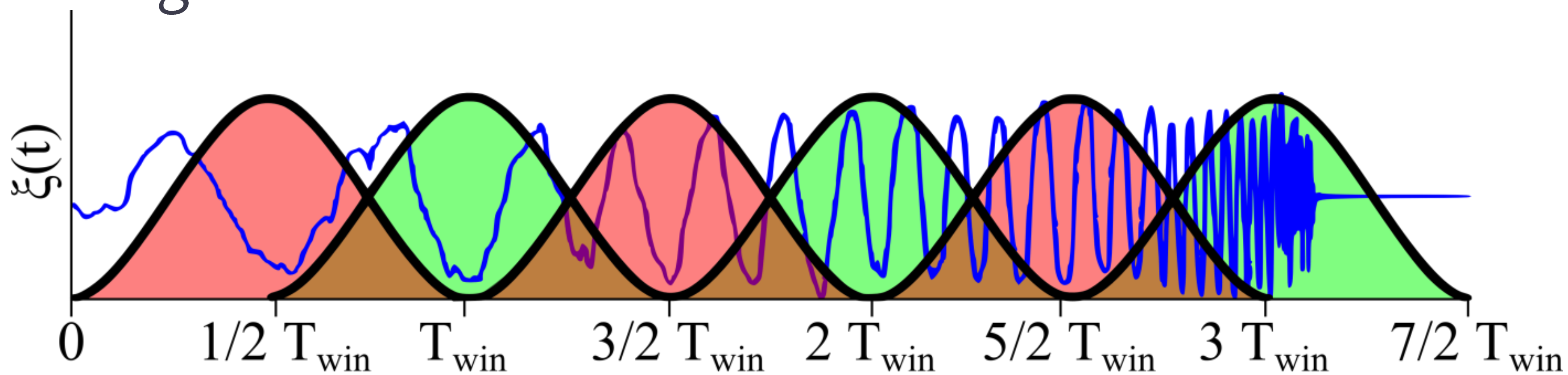
Frequency Responses Analysis

- Frequency response
  - Complex function given as a curve as a function of the excitation frequency, e.g.
    - Amplitude and phase characteristics
    - Real and imaginary characteristics
    - Amplitude-phase characteristics
    - ...
- Describes the dynamic system under study using an equivalent linear system that best reflects the relationship between the input and the output
- It does not require preliminary knowledge of the mathematical model of the tested object

- It can be used to identify unstable systems

Frequency Responses Analysis (SISO)

1. Remove bias and drift, then combine maneuvers into one record T_F
2. Filter data – elimination of potential aliasing of high-frequency noise
3. Divide the signal ξ(t) into mutually overlapping windows w(t) of width T_{win} - reducing random error

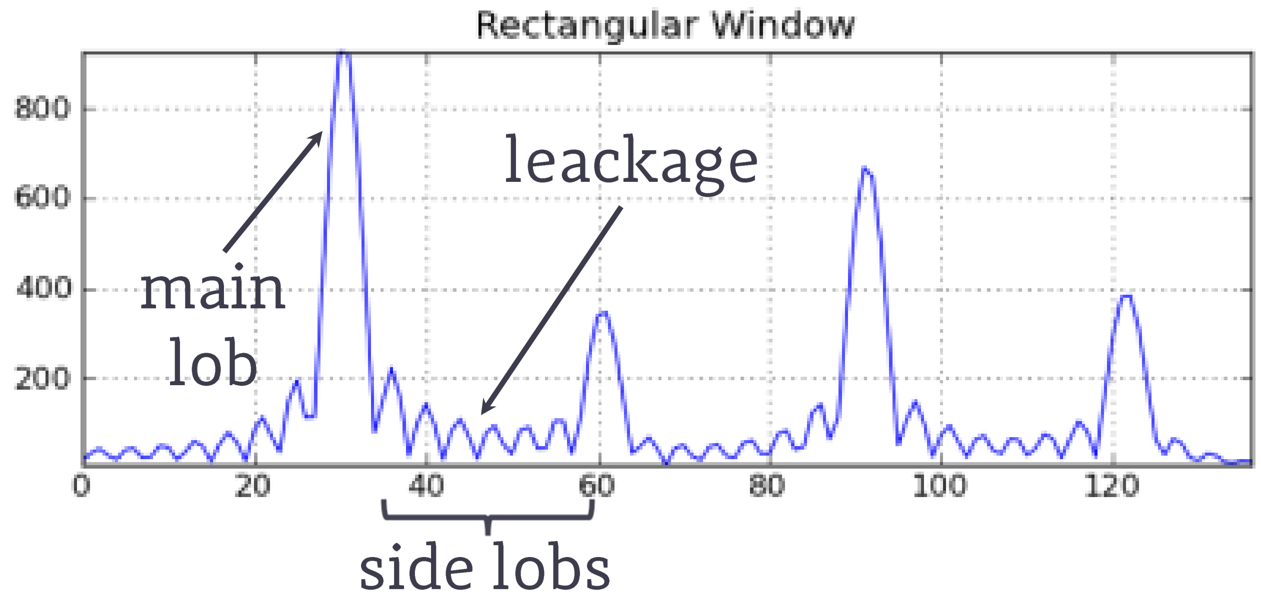


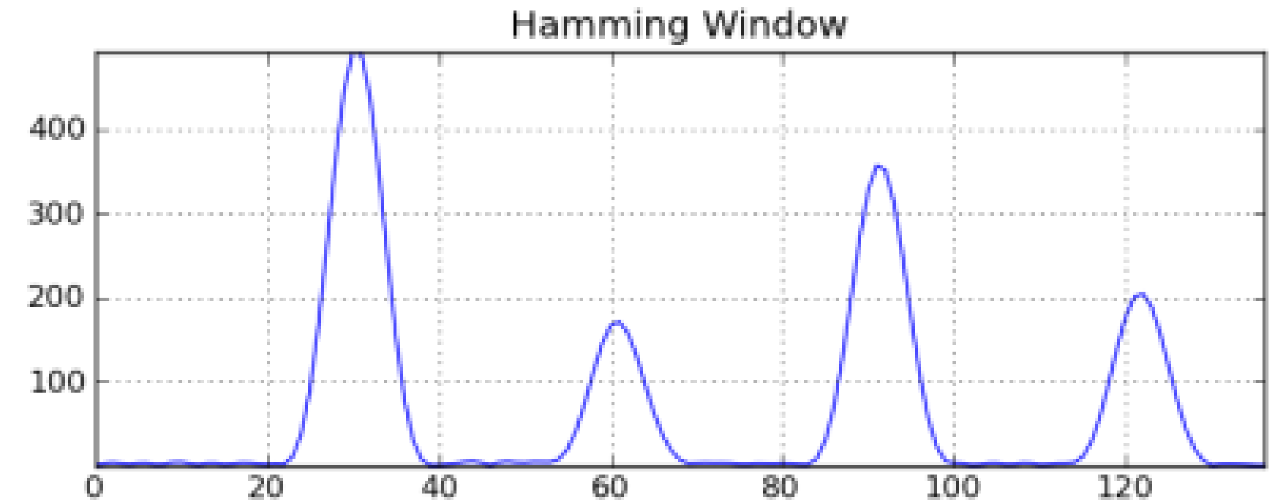
a. Calculate weighted response ξ(t)·w(t) (in each segment)
b. Obtain the transmittannce of the output and input signals (in each segment) using Chirp-Z transform
c. Calculate rough estimates of the power spectral density in each segment
d. Calculate smoothed power spectral density estimates over the entire time range (based on rough PSD estimates)

4. Calculate transfer functions and coherence

Frequency Responses Analysis (SISO)

- Smoothing windows – reduce errors due to spectra leackage (typical effect for rectangular window)





- Half-sine window – allows you to increase the accuracy of the obtained frequency response when compared to usually used Hanning window

$$w(t) = \sin\left(\pi \frac{t}{T_{win}}\right)$$

- 50% window overlap reduces random error by 26%
  - Further increasing the overlap increases the computational cost and slightly reduces the error (29% for 80% window overlap)

Frequency Responses Analysis (SISO)

- Window width selection
  - At least two segments $T_{win}=0.5T_{rec}$ should be used to analyze the shortest maneuver (that lasts $0.5T_{rec}$).
  - For longer windows, more accurate low-frequency estimates are obtained, but random error increases
    - for $T_{win}=1/5T_F$ error less than 20% in amplitude and 11.5deg in phase

      $T_{win} \leq \min\{0.5T_{rec} \quad 1/5T_F\}$
  - The shortest time window should contain at least one decade of bandwidth between f_{min} and f_{max}

    $T_{win} \geq 20/f_{max}$

Frequency Responses Analysis (SISO)

- Evaluating power spectra density rough estimates (in each segment)

$$\tilde{S}_{xx}(f) = \frac{2}{T}|X(f)|^2 \qquad \tilde{S}_{yy}(f) = \frac{2}{T}|Y(f)|^2 \qquad \tilde{S}_{xy}(f) = \frac{2}{T}\left|X^{\dagger}(f)Y(f)\right|$$

- Smoothed power spectral density estimates for the entire range (averaging PSD rough estimates)

$$\hat{S}_{xx}(f) = \left(\frac{1}{Un_r}\right)\sum_{k=1}^{n_r}\tilde{S}_{xx,k}(f)$$

n_r – numer of segments
U – coffecrtion factor
(for Half-sine window U=0.707)

$$\hat{S}_{yy}(f) = \left(\frac{1}{Un_r}\right)\sum_{k=1}^{n_r}\tilde{S}_{yy,k}(f)$$

$$\hat{S}_{yy}(f) = \left(\frac{1}{Un_r}\right)\sum_{k=1}^{n_r}\tilde{S}_{yy,k}(f)$$

- Frequency response and coherence

$$\hat{H}(f) = \frac{\hat{S}_{xy}(f)}{\hat{S}_{xx}(f)} \qquad \hat{\gamma}_{xy}^2(f) = \frac{\left|\hat{S}_{xy}(f)\right|^2}{\left|\hat{S}_{xx}(f)\right|\left|\hat{S}_{yy}(f)\right|}$$

Frequency Responses Analysis (MISO)

1. Evaluate matrix of cross spectra between each inputs and single output and matrix of auto- and cross spectra between each inputs
2. Power density matrix correction using conditioned coherence (removing correlation between inputs)

$$\gamma^2_{x_i y \cdot x_j}(\mathrm{f}) = \frac{\left|\hat{\mathbf{S}}_{x_i y \cdot x_j}(\mathrm{f})\right|^2}{\left|\hat{\mathbf{S}}_{x_i x_i \cdot x_j}(\mathrm{f})\right| \left|\hat{\mathbf{S}}_{yy \cdot x_j}(\mathrm{f})\right|}$$

x_i – i-th inputs signal
x_j – j-th output signal (i≠j)

- Conditioned random error

$$\varepsilon_{x_j}(\mathrm{f}) = C_\varepsilon \frac{\sqrt{1 - \gamma^2_{x_i y \cdot x_j}(\mathrm{f})}}{\left|1 - \gamma^2_{x_i y \cdot x_j}(\mathrm{f})\right| \sqrt{2 n_d}}$$

C_ε – window overlap constant
C_ε=0.74 for 50% overlap
n_d – numer of independednt time windows $n_d = T_{rec}/T_{win}$

3. Transfer functions estimation: $\hat{\mathbf{H}}(\mathrm{f}) = \hat{\mathbf{S}}^{-1}_{XX}(\mathrm{f}) \hat{\mathbf{S}}_{Xy}(\mathrm{f})$

- Frequency response analysis (MIMO)

1. Perform MISO analysis for each output signal
2. Selects the power spectrum calculated for the primary input
   - For the secondary input, estimation is perfomed for the data obtained from another maneuver

Frequency Responses Analysis

- Composite windowing (combining time windows of different width)
  - Shorter windows - improved accuracy for high frequencies
  - Longer windows - improved accuracy for low frequencies

  - Combining of time windows of different widths allows to obtain accurate frequency response at low and high frequencies
    - Weighting factors evaluated for the i-th window and each frequency
      $$W_i = \left[\frac{\varepsilon_i(f)}{\varepsilon_{min}(f)}\right]^{-4}$$
    - Evaluation of composite power spectra and composite coherence estimates, e.g.
      $$\hat{S}_{xx}(f) = \frac{\sum_{i=1}^{n_w} W_i^2(f)\,\hat{S}_{xx}(f)}{\sum_{i=1}^{n_w} W_i^2(f)} \qquad \hat{\gamma}_{xy}^2(f) = \frac{\left|\hat{S}_{xy}(f)\right|^2}{\left|\hat{S}_{xx}(f)\right|\left|\hat{S}_{yy}(f)\right|}$$

n_w – numer of time windows

Frequency Responses Analysis

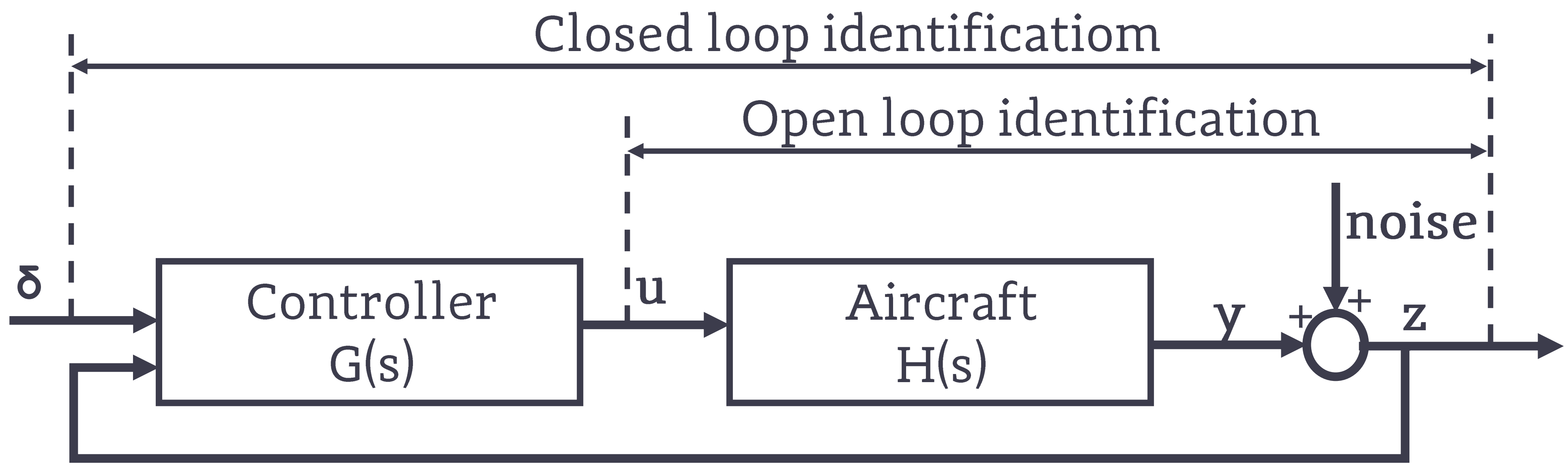
- Composite windowing (combining time windows of different width)
  - The final power density estimates minimize the cost function for each discrete frequency

$$J(f) = \sum_{i=1}^{n_w} W_i \left\{ \left(\frac{\hat{S}_{xx_c} - \hat{S}_{xx_i}}{\hat{S}_{xx}} \right)^2 + \left(\frac{\hat{S}_{yy_c} - \hat{S}_{yy_i}}{\hat{S}_{yy}} \right)^2 + \left[\frac{\mathrm{Re}(\hat{S}_{xy_c}) - \mathrm{Re}(\hat{S}_{xy_i})}{\mathrm{Re}(\hat{S}_{xy})} \right]^2 + \left[\frac{\mathrm{Im}(\hat{S}_{xy_c}) - \mathrm{Im}(\hat{S}_{xy_i})}{\mathrm{Im}(\hat{S}_{xy})} \right]^2 + 5 \left(\frac{\hat{\gamma}^2_{xy_c} - \hat{\gamma}^2_{xy_i}}{\hat{\gamma}^2_{xy}} \right)^2 \right\}$$

  - The cost function is non-linear
    - Optimization algorithm, e.g. Newton-Rhapson
      - Composite power density estimates as initial values
  - Final estimates of frequency response and coherence

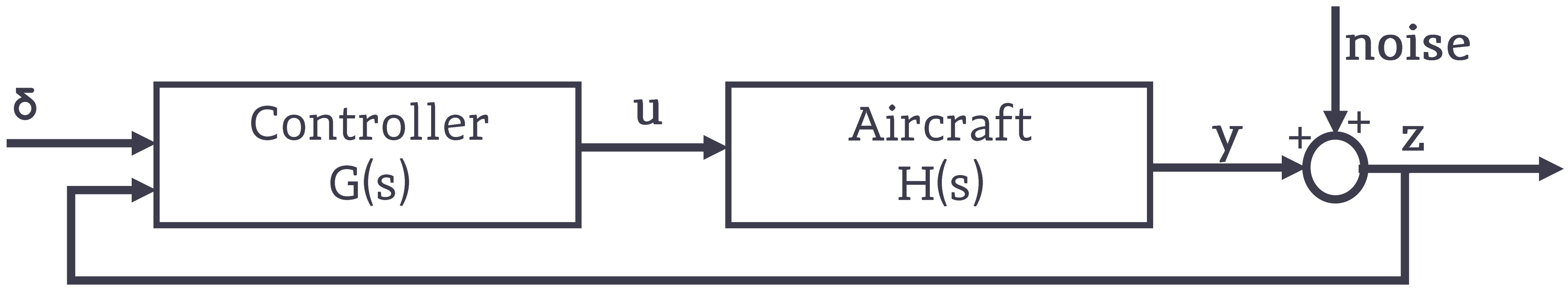
$$\hat{H}_c(f) = \frac{\hat{S}_{xy_c}(f)}{\hat{S}_{xx_c}(f)} \qquad \hat{\gamma}^2_{xy_c}(f) = \frac{\left|\hat{S}_{xy_c}(f)\right|^2}{\left|\hat{S}_{xx_c}(f)\right|\left|\hat{S}_{yy_c}(f)\right|}$$

Unstable aircraft identification



- A controller is used to improve stability of an unstable aircraft
- Closed-loop identification
  - Allows to obtain information about the aircraft with the controller
  - The system is stable - no numerical problems due to integration
- Open-loop identification
  - Allows to verify wind tunnel data and analytical predictions
- Provides more accurate models of the aircraft in order to modify stability augmentation systems

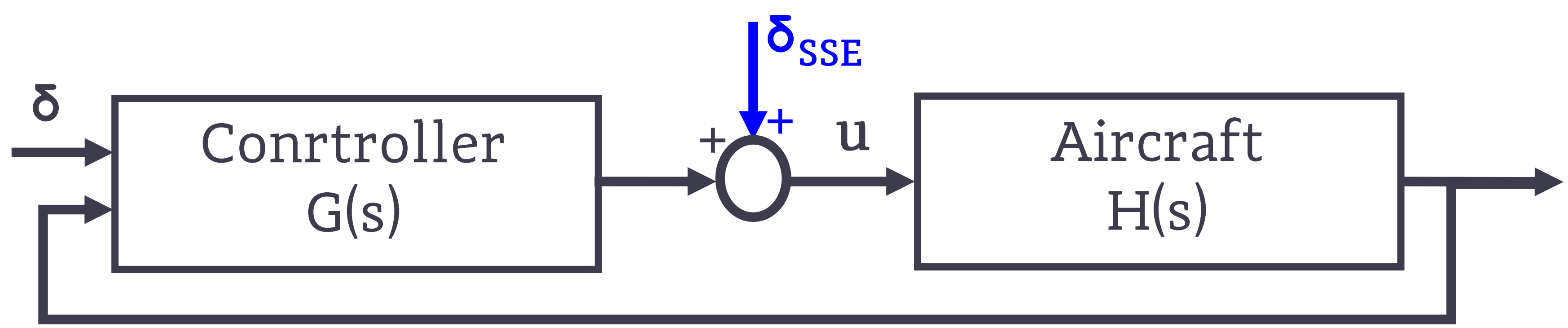
Unstable aircraft identification



- When the controller is known, it is possible to identify the aircraft in an open loop by using a closed loop
  - This approach is impractical
    - It requires detailed knowledge of the controller control laws and the dynamics of the control system elements
    - By suppressing motion, the controller drastically limits the amount of information contained in the measurement data (therefore, it reduces the accuracy of the estimated parameters)
    - Feedback leads to correlation between input signals and variables describing the aircraft motion
    - The aircraft response passes through the controller and thus the measurement noise present in it, acts as a stochastic input signal

Unstable aircraft identification

- Introducing controller causes correlation of independent variables – possible problems with identification
  - Changing the model structure so it contains fewer parameters
  - Using a-priori knowledge for model parameters
  - Planning experiments that allows for better excitation of the aircraft (conveying more information about the system)
    - Modifying the on-board computer to apply inputs by bypassing the stability augmentation system, e.g. directly to a given control surface



- The stability augmentation system remains active and after a certain time it will dampen the motion, however the initial response of the object is different than if the forces were applied in a conventional way

Unstable aircraft identification

- Methods:
  - Equation error – do not require integration, but problems due to data collinearity (due to feedback)
  - Output error – integration problems (time domain)
    - Combined Output Error and Least Squares Method
    - Equation decoupling Method
    - Eigenvalue Transformation Method
    - Output Error with Artificial Stabilization
    - Output Error Method in the Frequency Domain
    - Multiple Shooting Method
- Filter error method – Kalman gain matrix stabilizes numerical solution

Combined Output Error & Least Squares

- Measurements of state variables that cause instability x_m are used (as in the Least Squares Method) instead of integrating those variables (as in the Output Error Method)
  - No need to integrate state variables that cause integration problems

$$\dot{\mathbf{x}} = \mathbf{A}_S\mathbf{x}(t) + [\mathbf{B} \quad \mathbf{A}_U]\begin{bmatrix}\mathbf{u}(t) \\ \mathbf{x}_m(t)\end{bmatrix}$$

$$\mathbf{y} = \mathbf{C}_S\mathbf{x}(t) + [\mathbf{D} \quad \mathbf{C}_U]\begin{bmatrix}\mathbf{u}(t) \\ \mathbf{x}_m(t)\end{bmatrix}$$

$\mathbf{A}_S$ – state matrix with parameters corresponding to stable state variables
$\mathbf{A}_U$ – state matrix with parameters corresponding to unstable state variables

- The number of state variables causing instability is usually smaller than the number of other state variables
  - The method works more like the Output Error Method than the Least Squares Method

Equation decoupling method

- Using measurements of the state variables that cause instability x_m
- Separating equations to integrate each state equation separately

$$\dot{\mathbf{x}} = \mathbf{A}_D\mathbf{x}(t) + [\mathbf{B} \quad \mathbf{A}_{OD}]\begin{bmatrix}\mathbf{u}(t)\\ \mathbf{x}_m(t)\end{bmatrix}$$

$$\mathbf{y} = \mathbf{C}_D\mathbf{x}(t) + [\mathbf{D} \quad \mathbf{C}_{OD}]\begin{bmatrix}\mathbf{u}(t)\\ \mathbf{x}_m(t)\end{bmatrix}$$

$\mathbf{A}_D$ – state matrix containing on-diagonal elements of the state matrix $\mathbf{A}$

$\mathbf{A}_{OD}$ – state matrix containing off-diagonal elements of the state matrix $\mathbf{A}$

- Measurements of state variables that cause instability are subject to measurement noise
  - The matrix with off-diagonal $\mathbf{A}_{OD}$ elements introduces process noise
    - The method may cause numerical problems or affect the accuracy of the estimates unless the measurement noise is small

Equation decoupling method

- Example

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\\ b_{41} & b_{42}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}; \quad \boldsymbol{y}=\boldsymbol{x}; \quad \boldsymbol{z}=\boldsymbol{y}+\boldsymbol{\nu}$$

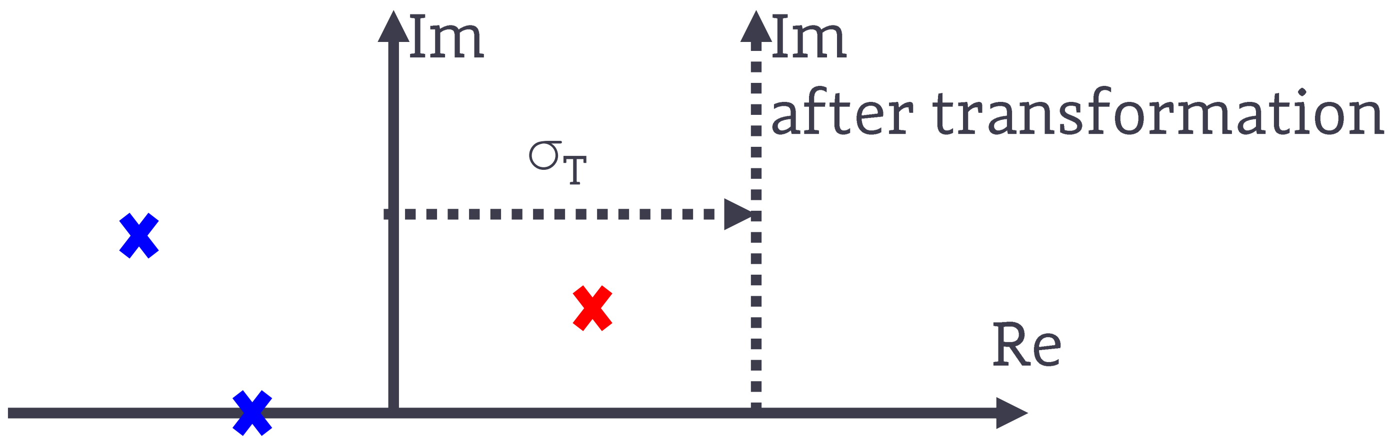
- Thus

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}\begin{bmatrix}a_{11} & 0 & 0 & 0\\ 0 & a_{22} & 0 & 0\\ 0 & 0 & a_{33} & 0\\ 0 & 0 & 0 & a_{44}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\\ b_{41} & b_{42}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}+\begin{bmatrix}0 & a_{12} & a_{13} & a_{14}\\ a_{21} & 0 & a_{23} & a_{24}\\ a_{31} & a_{32} & 0 & a_{34}\\ a_{41} & a_{42} & a_{43} & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\\ z_4\end{bmatrix}$$

- Measurements of state variables that cause instability are subject to measurement noise
  - The matrix with off-diagonal $\mathbf{A}_{\mathrm{OD}}$ elements introduces process noise
    - The method may cause numerical problems or affect the accuracy of the estimates unless the measurement noise is small

Eigenvalue Transformation Method

- Converting an unstable aircraft probleminto a stable one by using a transformation in the complex plane (shifting the complex axis by σ_T)



$$\tilde{\mathbf{x}}(t) = \mathbf{x}(t)\exp(-\sigma_T t)$$
$$\tilde{\mathbf{y}}(t) = \mathbf{y}(t)\exp(-\sigma_T t)$$
$$\tilde{\mathbf{u}}(t) = \mathbf{u}(t)\exp(-\sigma_T t)$$
$$\tilde{\mathbf{z}}(t) = \mathbf{z}(t)\exp(-\sigma_T t)$$
$$\tilde{\mathbf{w}}(t) = \mathbf{w}(t)\exp(-\sigma_T t)$$
$$\tilde{\mathbf{v}}(t) = \mathbf{v}(t)\exp(-\sigma_T t)$$

  - After transformation

$$\dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A} - \sigma_T \mathbf{I})\tilde{\mathbf{x}}(t) + \mathbf{B}\tilde{\mathbf{u}}(t) + \mathbf{F}\tilde{\mathbf{w}}(t)$$

$$\tilde{\mathbf{y}}(t) = \mathbf{C}\tilde{\mathbf{x}}(t) + \mathbf{D}\tilde{\mathbf{u}}(t) \qquad \tilde{\mathbf{z}}(t_k) = \tilde{\mathbf{y}}(t_k) + \mathbf{G}\tilde{\mathbf{v}}(t_k)$$

- Only the diagonal elements of **A** are changed by σ_T
  - If the state matrix **A** elements appear in the output matrix **C**, it is necessary to perform inverse transform before calculating outputs **y**
- The function $\exp(-\sigma_T t)$ is decreasing - the transformed variables tend to zero for long maneuvers
  - The more unstable the system, the shorter the maneuver

Output Error with Artificial Stabilization

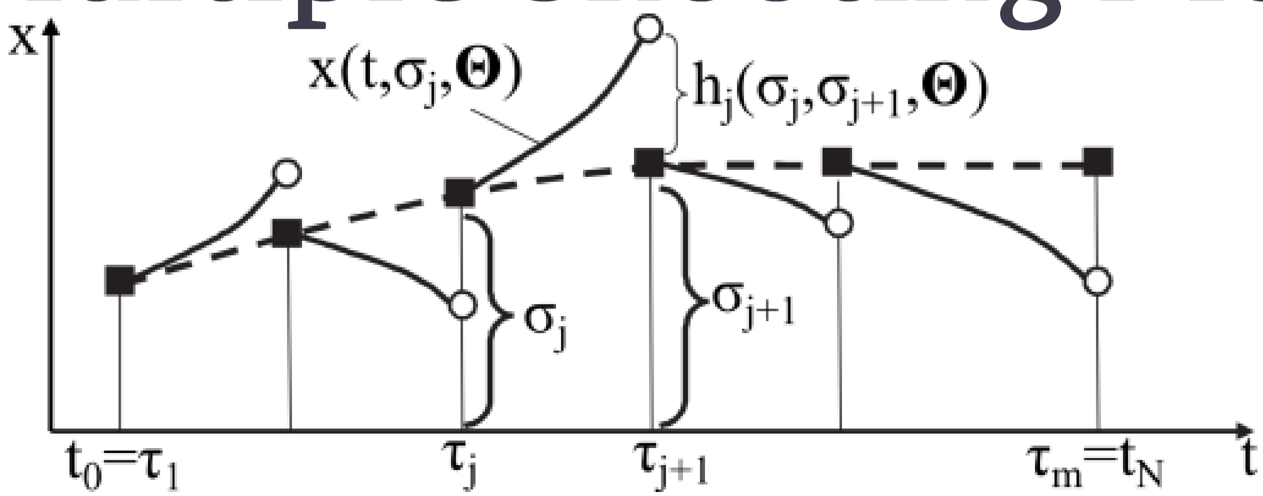
- Auxiliary matrix S is introduced to stabilize the solution

  $$\hat{\mathbf{x}}(t_k) = \mathbf{x}(t_k) + \mathbf{S}[\mathbf{z}(t_k) - \mathbf{y}(t_k)]$$

  - When the stabilization matrix S is a zero matrix, it is equivalent to the output error method
  - When the stabilization matrix is an identity matrix and only state variables are used as output variables, it is equivalent to the equation error method

- Inclusion of the stabilization matrix introduces a modeling error (reduces the accuracy of the estimates)
  - The modeling error will be small if the elements of the stabilization matrix are small

Multiple Shooting Method



- Dividing of the integration interval into subintervals
- Solutions to the problem within each sub-range

$$\dot{\mathbf{x}} = f(t, \mathbf{x}, \mathbf{\Theta}) \qquad t \in [\tau_j \quad \tau_{j+1}]$$
$$\mathbf{x}(\tau_j) = \boldsymbol{\sigma}_j$$

  - The initial conditions σ_j for each interval are also unknown - they should be included as estimates
  - Moreover, the continuity condition must be met

$$h_j(\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_{j+1}, \mathbf{\Theta}) = \mathbf{x}(\tau_{j+1} | \boldsymbol{\sigma}_j, \mathbf{\Theta}) - \boldsymbol{\sigma}_{j+1} \qquad j = 1, \ldots, m-1$$

- The multi-shot method is not the same as the output error method for several data sets, because the conventional output error method does not assume continuity between subsequent data sets.

Flight Path Reconstruction

- Checking whether the measurements correspond to each other and removing systematic errors
- Obtaining accurate values for state variables from other flight parameters

$$\dot{U} = -QW + RV - g\sin\Theta + a_x$$
$$\dot{V} = -RU + PW + g\cos\Theta\sin\Phi + a_y$$
$$\dot{W} = -PV + QU + g\cos\Theta\cos\Phi + a_z$$
$$\dot{\Phi} = P + Q\sin\Phi\tan\Theta + R\cos\Phi\tan\Theta$$
$$\dot{\Theta} = Q\cos\Phi - R\sin\Phi$$
$$\dot{\Psi} = Q\sin\Phi\sec\Theta + R\cos\Phi\sec\Theta$$
$$\dot{h} = U\sin\Theta - V\cos\Phi\sin\Theta - W\cos\Phi\cos\Theta$$

- Accelerations and angular velocities are usually measured with high accuracy (unlike e.g. flow angles)
  - Based on a_x, a_y, a_z and P, Q, R: the U, V, W; ϕ, Θ, ψ and h are estimated along with the initial conditions (i.e. U_0,V_0, ...)

Flight Path Reconstruction

- After estimating the state variables (e.g. using the output error method), it is possible to estimate sensor models

  $$y_m = K_y y(t - \tau_y) + b_y$$

- This mainly applies to aerodynamic sensors and requires the estimation of scale factor, biases and time delays
- For linear accelerations and angular velocities
  - Mounting errors are smaller than measurement noise - can be neglected
  - Scale errors are usually small and highly correlated with biases - should be ignored
  - Time delays are small - can be ignored
  - Biases should not be omitted to avoid drift when integrating the equations of motion
  - The sensors location should be taken into account

    $$\mathbf{a}_C = \mathbf{a}_O + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{OC}) + \boldsymbol{\varepsilon} \times \mathbf{r}_{OC}$$

Flight Path Reconstruction

- Five-hole tube

$$V_C = \sqrt{U^2 + V^2 + W^2}$$

  - The pneumometric sensor is placed on the boom in front of the object
    - The CG offset must be taken into account

$$\mathbf{V}_{NB} = \mathbf{V}_C + \mathbf{\Omega} \times \mathbf{r}_{C\ NB}$$

      - If the boom is too short, the scale factors may depend on the aircraft configuration

$$\alpha_{NB} = K_\alpha \mathrm{atan}\left(\frac{W_{NB}(t-\tau_\alpha)}{U_{NB}(t-\tau_\alpha)}\right) + b_\alpha$$

$$\beta_{NB} = K_\beta \mathrm{asin}\left(\frac{V_{NB}(t-\tau_\alpha)}{\sqrt{U_{NB}^2(t-\tau_\alpha) + V_{NB}^2(t-\tau_\alpha) + W_{NB}^2(t-\tau_\alpha)}}\right) + b_\beta$$

Flight Path Reconstruction

- In the FPR a_x, a_y, a_z and P, Q, R were equivalent to the input signals
  - (Those) flight parameters measurements are subject to measurement noise
    - Thus, in the estimation problem, there is noise in the input signals
    - Therefore, the process is stochastic
      - When high-quality measurement sensors are used, this noise is small - the process can be treated as deterministic
      - An extended Kalman filter can be used to include noise in the input signals

- An alternative to motion path reconstruction is the process of: Estimation before modeling

Estimation Before Modeling

- A two-stage process in which, in the first step, smoothed state variables time histories are obtained, and in the second step, model parameters are obtained by linear regression
  - Unlike FPR (also two-stage), linear accelerations and angular velocities are also estimated
    - Additionally, the normalized components of forces X, Y, Z and moments N, M, N acting on the object are estimated
  - This requires expanding the model by adding $\dot{P}$, $\dot{Q}$, $\dot{R}$

$$
\begin{aligned}
\dot{\mathrm{U}} &= -\mathrm{QW} + \mathrm{RV} - \mathrm{g}\sin\Theta + X \\
\dot{\mathrm{V}} &= -\mathrm{RU} + \mathrm{PW} + \mathrm{g}\cos\Theta\sin\Phi + Y \\
\dot{\mathrm{W}} &= -\mathrm{PV} + \mathrm{QU} + \mathrm{g}\cos\Theta\cos\Phi + Z \\
\dot{\Phi} &= \mathrm{P} + \mathrm{Q}\sin\Phi\tan\Theta + \mathrm{R}\cos\Phi\tan\Theta \\
\dot{\mathrm{P}} &= \mathrm{PQ}C_{11} + QRC_{12} + QC_{13} + L + NC_{14} \\
\dot{\mathrm{Q}} &= \mathrm{PR}C_{21} + (R^2 - P^2)C_{22} - RC_{23} + M \\
\dot{\mathrm{R}} &= \mathrm{PQ}C_{31} + QRC_{32} - QC_{33} + LC_{34} + N \\
\dot{\Theta} &= \mathrm{Q}\cos\Phi - \mathrm{R}\sin\Phi \\
\dot{\Psi} &= \mathrm{Q}\sin\Phi\sec\Theta + \mathrm{R}\cos\Phi\sec\Theta \\
\dot{\mathrm{h}} &= \mathrm{U}\sin\Theta - \mathrm{V}\cos\Phi\sin\Theta - \mathrm{W}\cos\Phi\cos\Theta
\end{aligned}
$$

Estimation Before Modeling

- Mathematical model

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \boldsymbol{\chi}(t), \boldsymbol{\Theta}) + \mathbf{w}$$

$$\dot{\boldsymbol{\chi}} = \mathbf{L}\boldsymbol{\chi}(t) + \boldsymbol{\xi}$$

$$\mathbf{y} = \mathbf{g}(\mathbf{x}(t), \boldsymbol{\Theta})$$

$$\mathbf{z}(t_k) = \mathbf{y}(t_k) + \mathbf{v}(t_k)$$

$$\chi = [X \quad Y \quad Z \quad L \quad M \quad N]^T$$

Gaussian noise with zero mean value ξ is obtained using a random number generator

- In this approach, forces and moments are not equivalent to the input signals, but are modeled as a third-order Gauss-Markov process

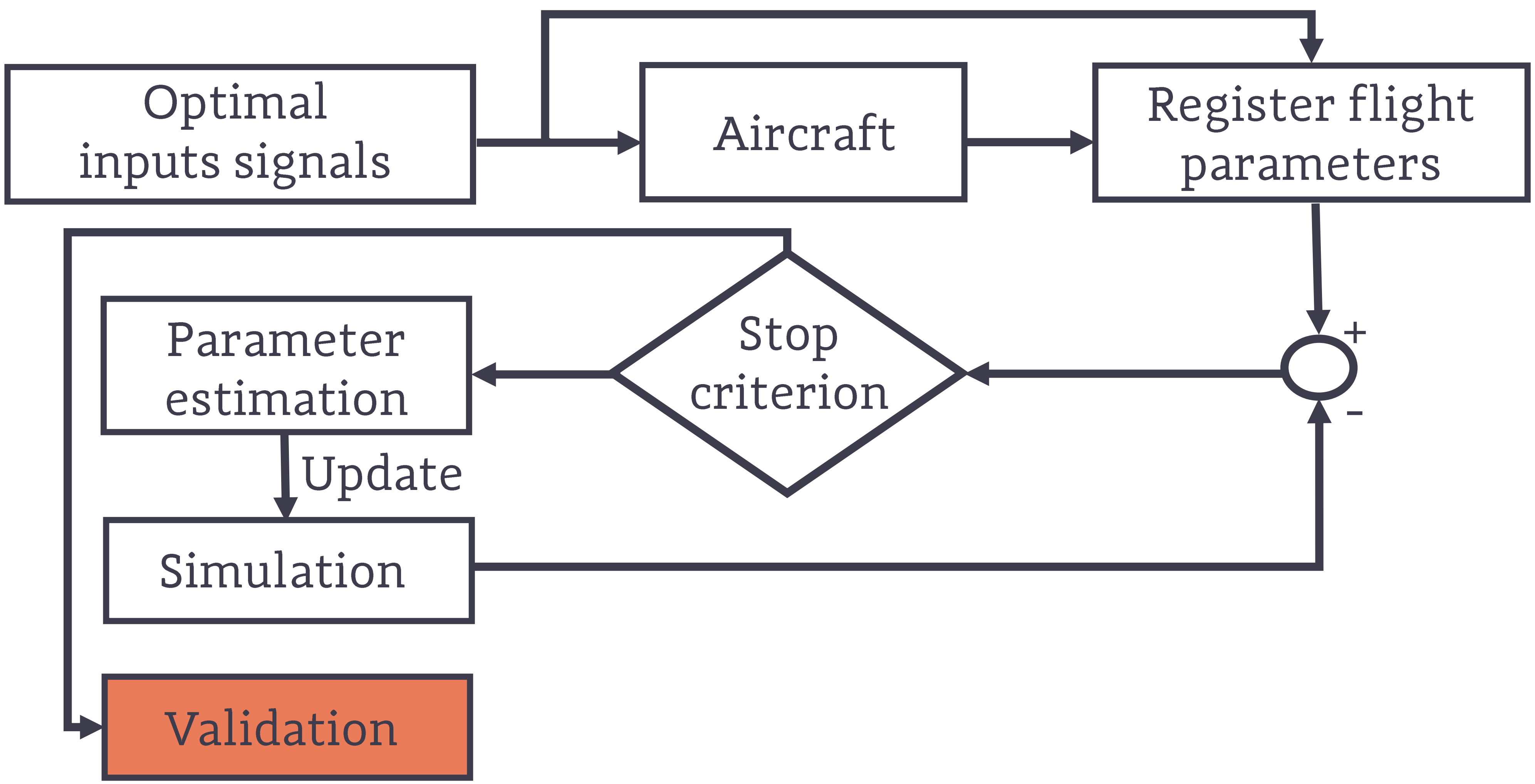
$$\dot{\chi}_i = L_i \chi_i + \xi_i(t) \quad i = 1, \dots, 6$$

$$L_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- The model includes 18 additional state variables. Increased computational complexity
  - Problem can be simplified due to the large number of zeros
- An extended Kalman filter is used to estimate sensor parameters and directly forces and moments (instead of stability and control derivatives).

Mathematical model

- Simple e.g. minimum number of model parameters
  - Convergence
  - Less time required for estimation
- Complex - to imitate all significant features of the object



Mathematical model

- Phenomenological – based on physical principles
  - Most popular
  - Parametric / Non-parametric
- Behavioral – maps the object response without any knowledge of underlying physical principles
  - Used in artificial neural networks

	Phenomenological model	Behavioral model
Parameters	Have physical meaning	Have no physical meaning
Simulation	Complex and difficult	Fast and easy
A-priori knowledge	Stored in the model	Not required
Applicability range	Large	Limited

- White box – phenomenological model
- Black box – behavioral model
- Grey box – combines white box and black box models

Mathematical model

- Parametric – model structure is strictly defined and contains model parameters
  - State-space models
    - Linear/nonlinear
    - Continuous/Discrete
    - Stochastic/Deterministic
    - Stationary/Nonstationary
  - Transfer function
- Non-parametric – model does not have a strictly defined structure, it is given
  as a curve, table
  - Impulse response, Step response
  - Frequency response
  - Energy spectral density

Validation

- Statistical properties of the estimates
  - Standard deviations
    - Relative standard deviations
  - Correlation coefficients
- Analysis of the residuals
  - Cost function
  - Thiel inequality coefficient
    - Error decomposition
  - Autocorrelation of the residuals
  - Power spectral density of the residuals
- Model predictive capabilities
  - Inverse simulation
  - Estimated model analysis
    - Comparing outcomes with results from other sources
    - Analysis of the outcomes physical sense (e.g. eigenvalues and eigenvectors)
  - Proof-of-match
  - Estimated model analysis in the frequency domain

Residuals covariance matrix

- Fisher Information Matrix Inverse

$$\mathbf{F} \approx \sum_{k=1}^{N} \left[\frac{\partial \mathbf{y}(t_k)}{\partial \mathbf{\Theta}}\right]^T \mathbf{R}^{-1} \left[\frac{\partial \mathbf{y}(t_k)}{\partial \mathbf{\Theta}}\right]$$

  - Standard deviations

$$\sigma_{\mathbf{\Theta}_i} = \sqrt{P_{ii}}$$

  - Correlation coefficients between estimates

$$\rho_{\mathbf{\Theta}_i \mathbf{\Theta}_j} = \frac{P_{ij}}{\sqrt{P_{ii} P_{jj}}}$$

  - Relative standard deviation

$$\sigma_{\mathrm{rel}\mathbf{\Theta}_i} = 100 \frac{\sigma_{\mathbf{\Theta}_i}}{\hat{\mathbf{\Theta}}_i}$$

- Not all assumptions are fully met (e.g. measurement noise is not Gaussian)
  - Estimates are too optimistic
    - Introduction of a correction factor of 5-10

Cost function

- A direct way to assess the model quality
- Difficulties in defining a direct criterion for various models obtained from identification
- Depends on the number of output signals and their units
- Too many parameters provide to much flexibility
- A low value of the cost function does not guarantee that all estimates are accurate (a significant improvement in the fit of one output signal may lead to a worse fit of the others if the overall cost function drops)
  - Output signals standard deviations allow for error detection

$$\sigma_i = \sqrt{\frac{1}{N}\sum_{k=1}^{N}[z_i(t_k) - y_i(t_k)]^2} \quad i = 1, \dots, n_y$$

- If multiple maneuvers are analyzed, this must be performed separately for each maneuver
  - Detection of poorly planned experiments
  - Wrong model structure and improper identification method

Theil Inequality Coefficient

- Greater emphasis on the correlation between measurement and model response (when compared to cost function)

$$U_i = \frac{\sqrt{\frac{1}{N}\sum_{k=1}^{N}[z_i(t_k) - y_i(t_k)]^2}}{\sqrt{\frac{1}{N}\sum_{k=1}^{N}[z_i(t_k)]^2} + \sqrt{\frac{1}{N}\sum_{k=1}^{N}[y_i(t_k)]^2}} \quad i = 1, \dots, n_y$$

  - Value 0 – measurement and model response are identical (ideal case)
  - Value 1 - The measurement and model response are completely different
  - In practice, values between 0.25 and 0.30 mean good model fit

- Thiel coefficient decomposition
  - Systematic error U^M
  - Nonsystematic error U^C
  - Ability to duplicate the variability in the true system U^S

Theil Inequality Coefficient

- Systematic error U^M

$$U_i^M = \frac{(\bar{z}_i - \bar{y}_i)^2}{\sqrt{\frac{1}{N}\sum_{k=1}^{N}[z_i(t_k)]^2} + \sqrt{\frac{1}{N}\sum_{k=1}^{N}[y_i(t_k)]^2}}$$

$$\sigma_{z_i} = \sqrt{\frac{1}{N}\sum_{k=1}^{N}[z_i(t_k) - \bar{z}_i]^2}$$

- Nonsystematic error U^C

$$U_i^C = \frac{2(1-\rho_i)\sigma_{z_i}\sigma_{y_i}}{\sqrt{\frac{1}{N}\sum_{k=1}^{N}[z_i(t_k)]^2} + \sqrt{\frac{1}{N}\sum_{k=1}^{N}[y_i(t_k)]^2}}$$

$$\sigma_{y_i} = \sqrt{\frac{1}{N}\sum_{k=1}^{N}[y_i(t_k) - \bar{y}_i]^2}$$

- Predicting capabilities U^S

$$\rho_i = \frac{1}{\sigma_{z_i}\sigma_{y_i}}\frac{1}{N}\sum_{k=1}^{N}[z_i(t_k) - \bar{z}_i][y_i(t_k) - \bar{y}_i]$$

$$U_i^S = \frac{\left(\sigma_{z_i} - \sigma_{y_i}\right)^2}{\sqrt{\frac{1}{N}\sum_{k=1}^{N}[z_i(t_k)]^2} + \sqrt{\frac{1}{N}\sum_{k=1}^{N}[y_i(t_k)]^2}}$$

  - The sum of the decomposition terms is 1
  - Large U^M and U^S values (above 0.1) mean that the model may require updating

Autocorrelation and PSD

- A statistical measure used to check whether the residuals are independent for different time points (whether they are described with white noise)

$$C_i(\tau) = \frac{1}{N}\sum_{k=\tau}^{N}[\upsilon(k) - \bar{\upsilon}][\upsilon(t_k - \tau) - \bar{\upsilon}]^T$$

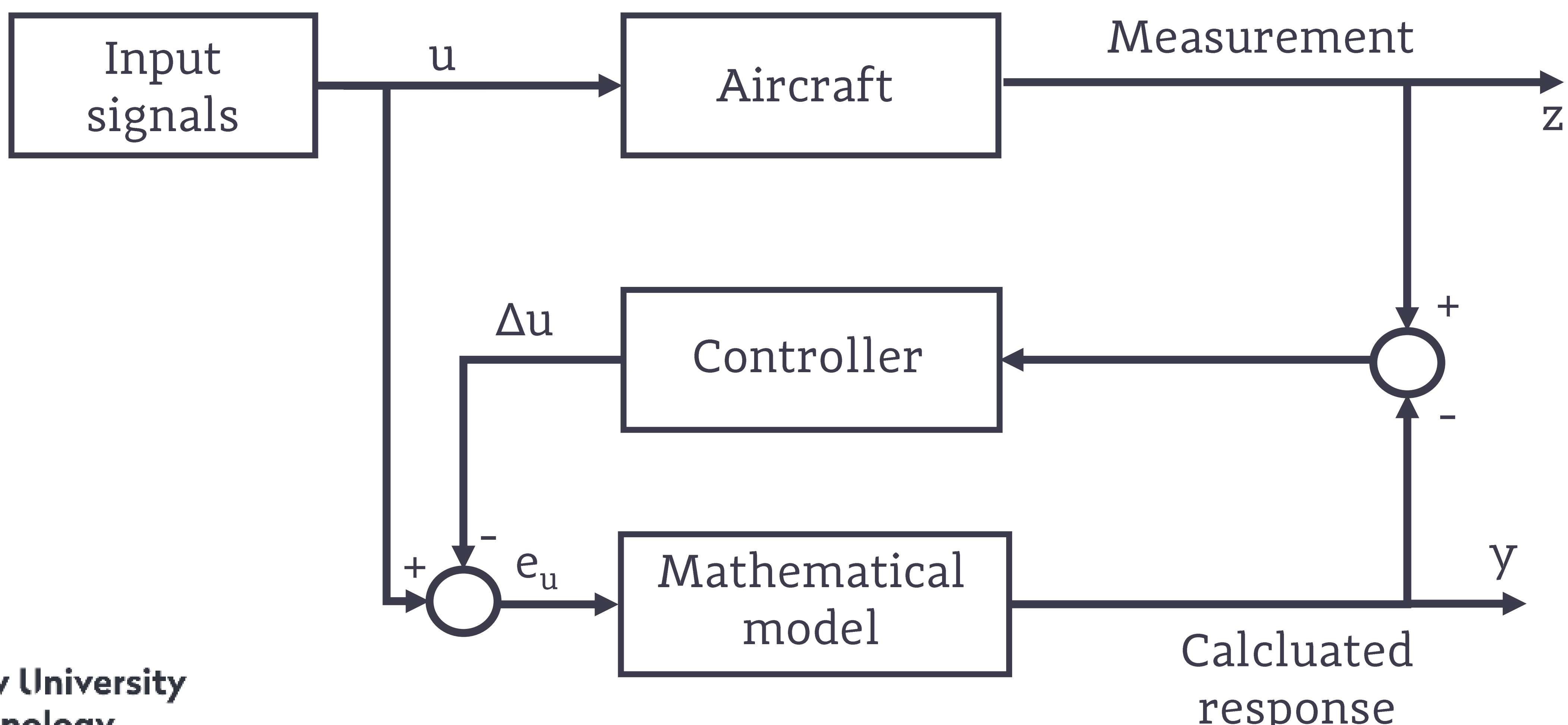
- When the residuals denote white noise, $C_i = 0$ for each τ
  - In practice, it is assumed that C_i for all τ above 1 should lie in the band $\pm 1.96/\sqrt{N}$ more than 95% of the times

- Power spectral density of the residuals
  - An alternative to autocorrelation of the residuals (test of whitness)
    - For white noise, the power spectrum is uniform over the entire frequency range

- Autocorrelation and Power spectral density of the residuals are used to check whether processing noise should be taken into account

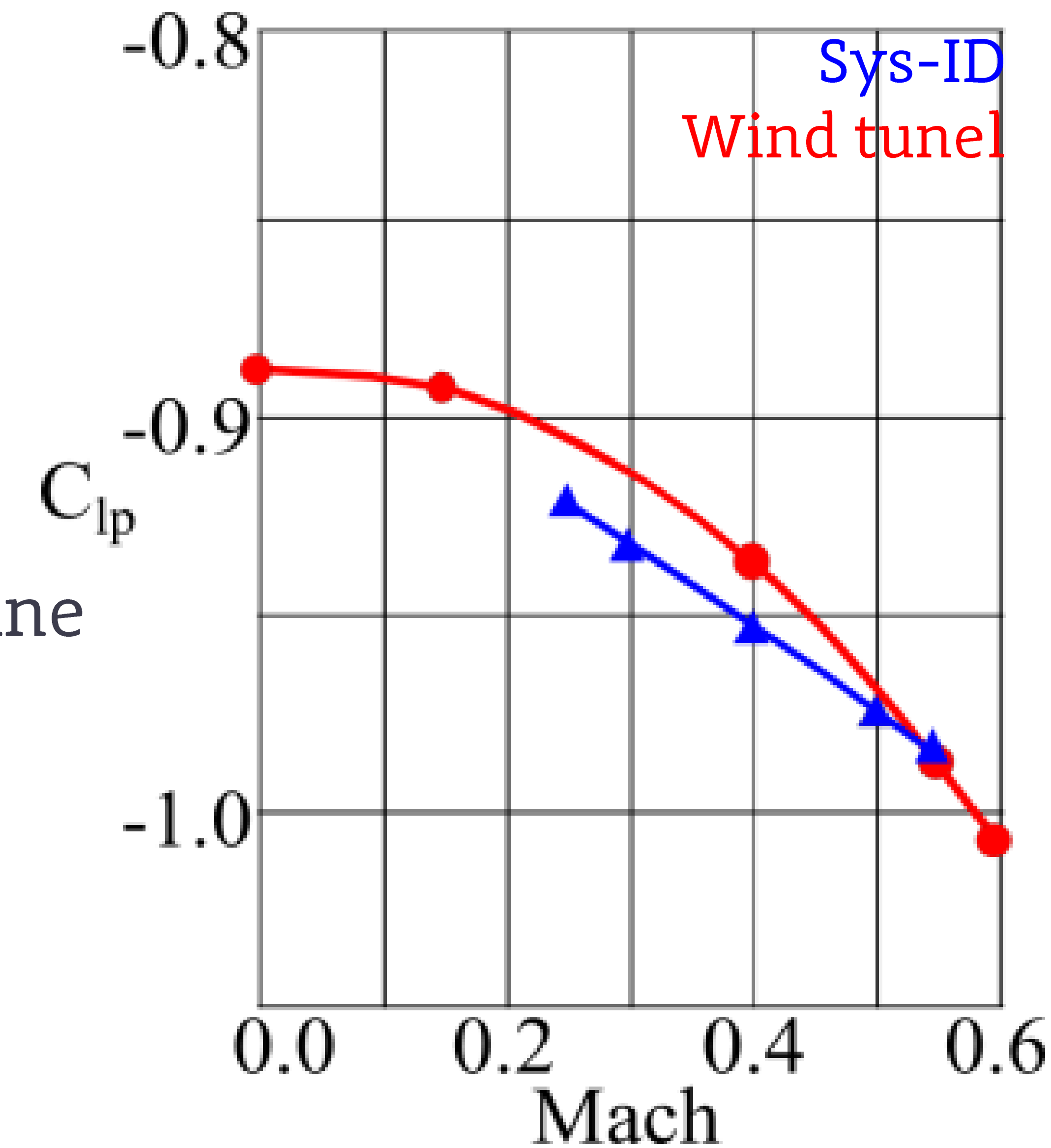
Inverse simulation

- Determining model quality using inputs instead of outputs
- The measured inputs u are used to obtain the model response y
  - Based on the model and controller responses, the control error e_u is calculated
    - The model is sufficiently accurate if the control error is less than 0.5deg



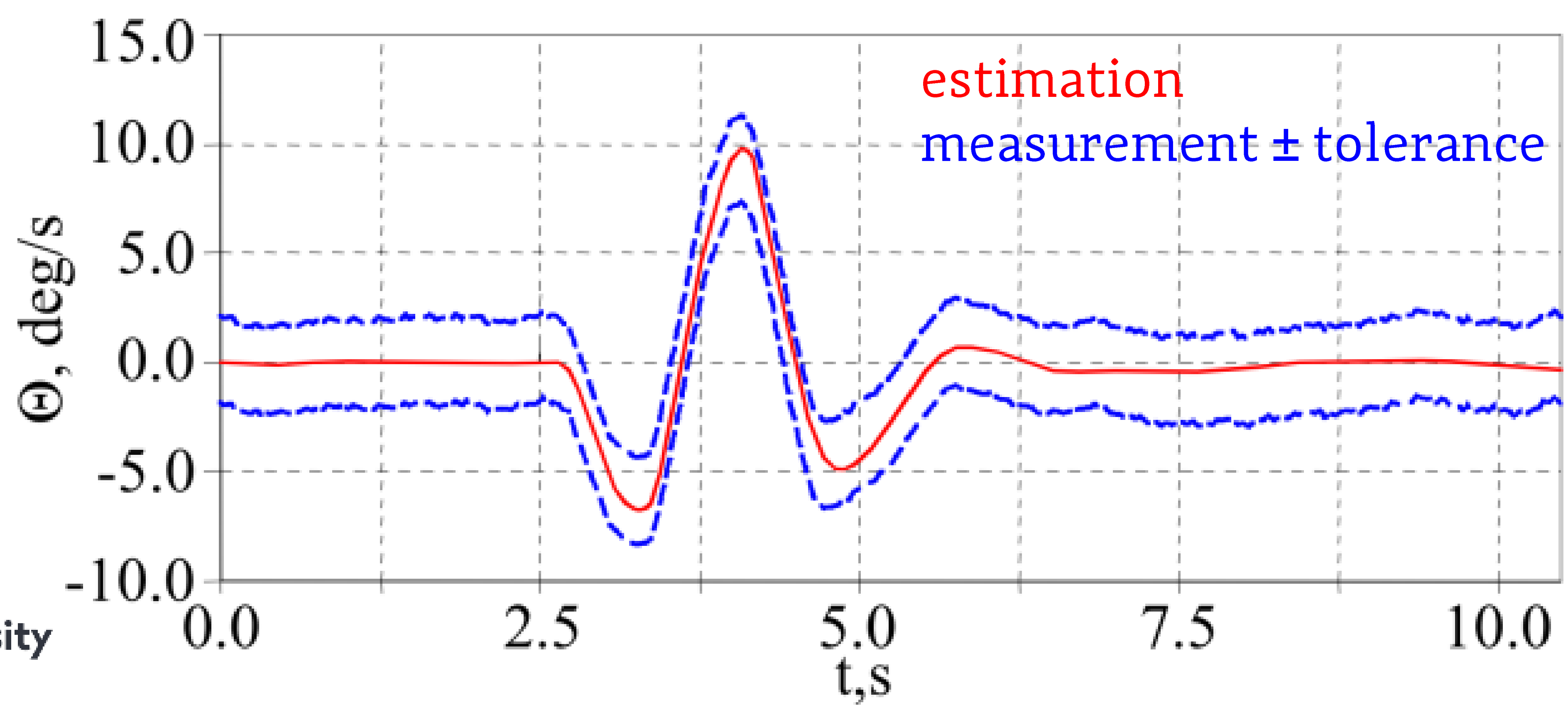
Comparing with other sources

- Comparison of estimates with values from other sources
  - Wind tunel tests
  - Analytical formulas
- Physical sense of the obtained results
  - Determining the influence of model parameters on its behavior and the physical relationship between these parameters.
    Response analysis in the complex plane for an equivalent system (LOES)
    - Eigenvalues inform about the motion type
    - Eigenvectors inform which state variables are dominant for a given motion type



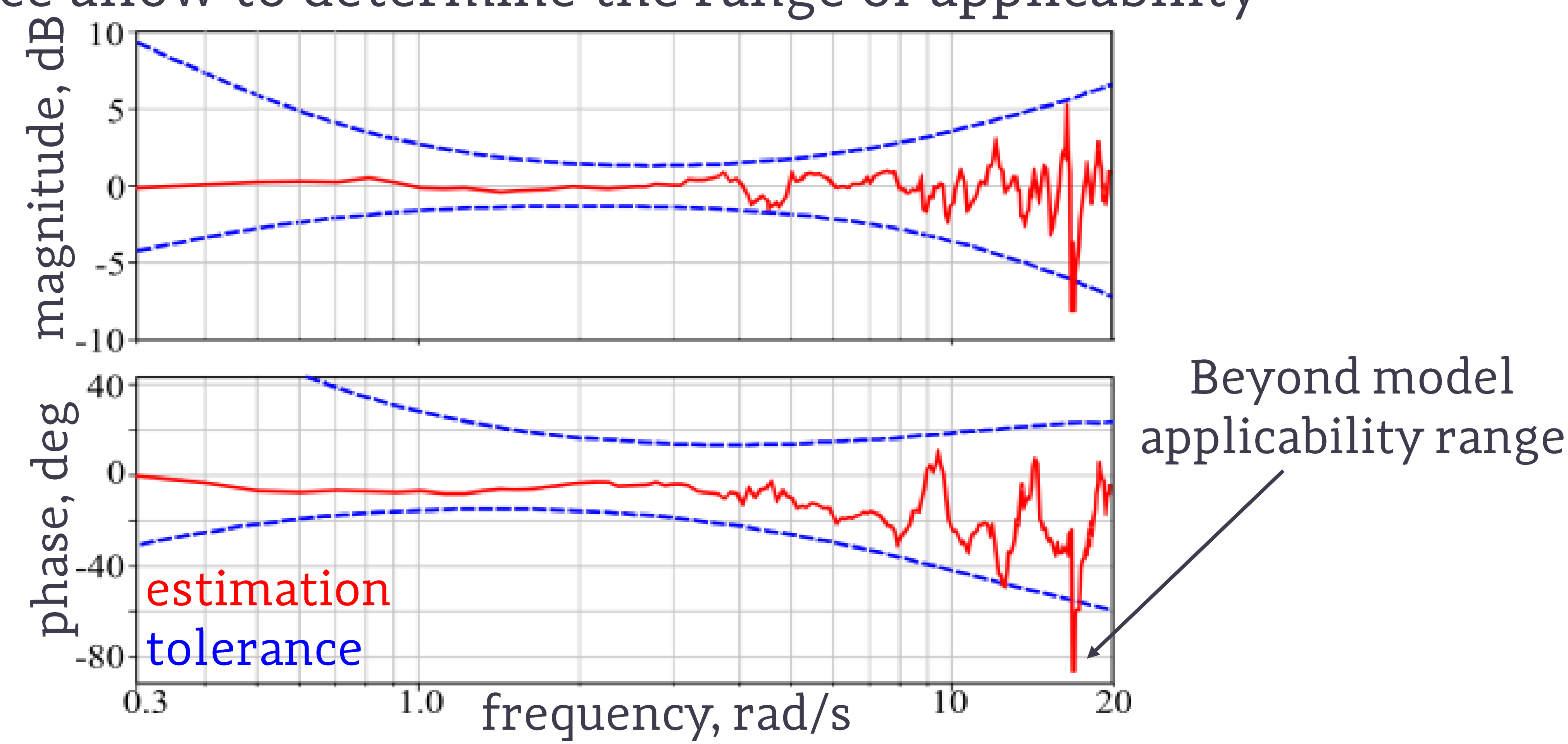
Proof-of-match

- Simulation of the aircraft response for data not used during the identification
- Comparisson between the calcualated outcomes and the measurement
  - Small differences indicate good predictive capabilities of the model
  - Error bounds determined e.g. from aeronautical regulations
  - There can be noise in input signals and flight parameter values at the trim point for the data set not used during identification
    - It is allowed to take biases into account for these quantities



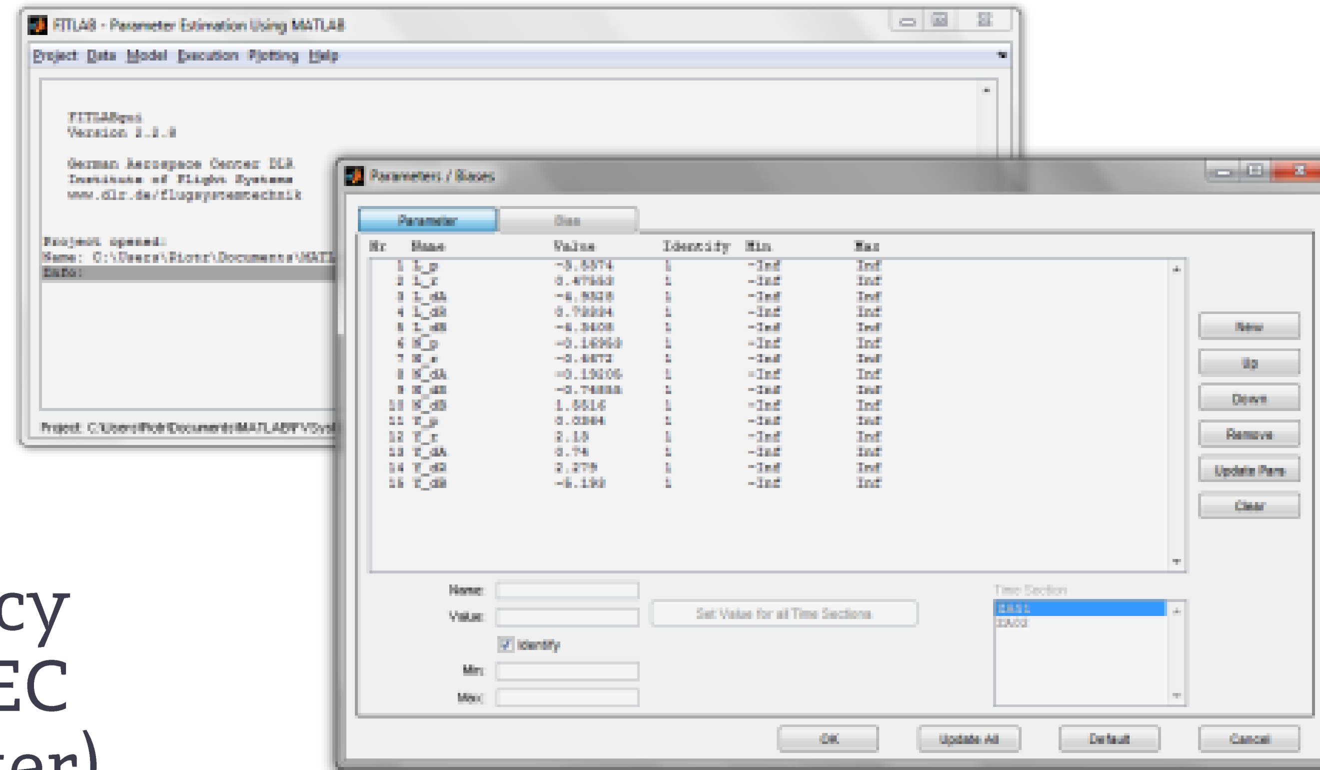
Proof-of-match in frequency domain

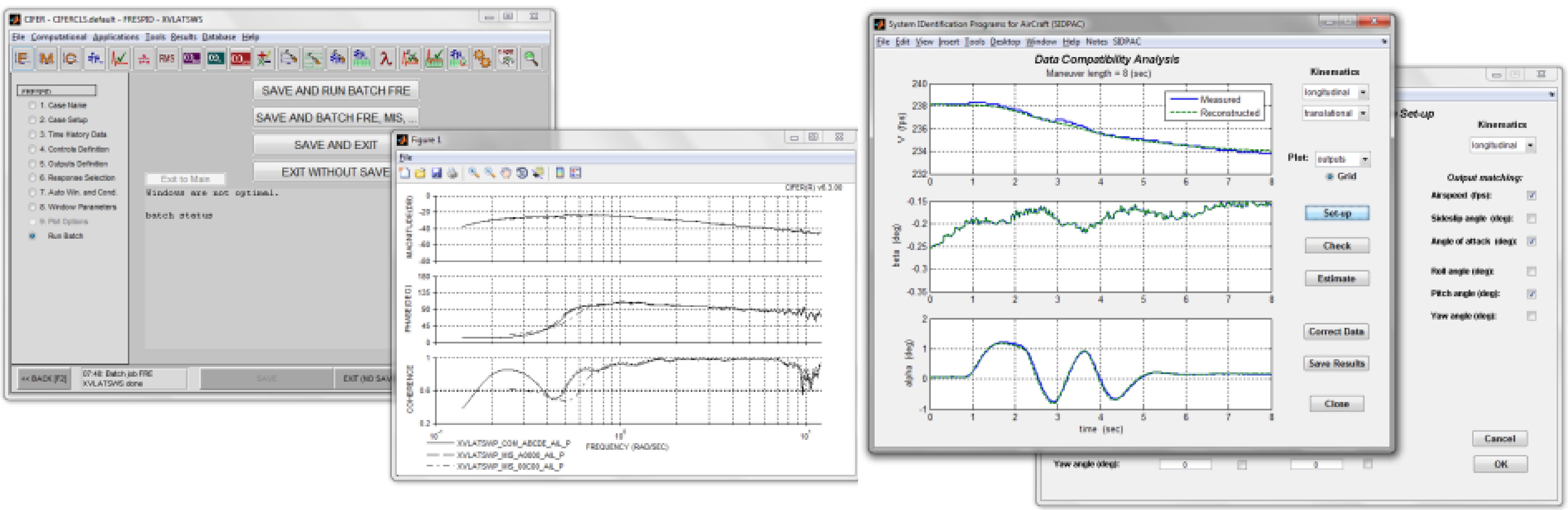
- Obtained for an equivalent system (LOES)
- In the ideal case, the differences in magnitude between measurement and the estimated model response is 0 dB, and the phase difference is 0 deg
- Model tolerances expressed in terms of magnitude and phase
  - Tolerance is determined based on the maximum dynamics of the system which when included will go unnoticed by the pilot (Maximum Unnoticeable Added Dynamics)
  - Tolerance allow to determine the range of applicability



Software

- FITLAB (DLR)
- SIDPAC - System IDentification Programs for AirCraft (NASA Langley Research Centre)
- CIFER - Comprehensive Identification form FrEquency Responses (US Army AMRDEC - NASA Ames Research Center)





Example

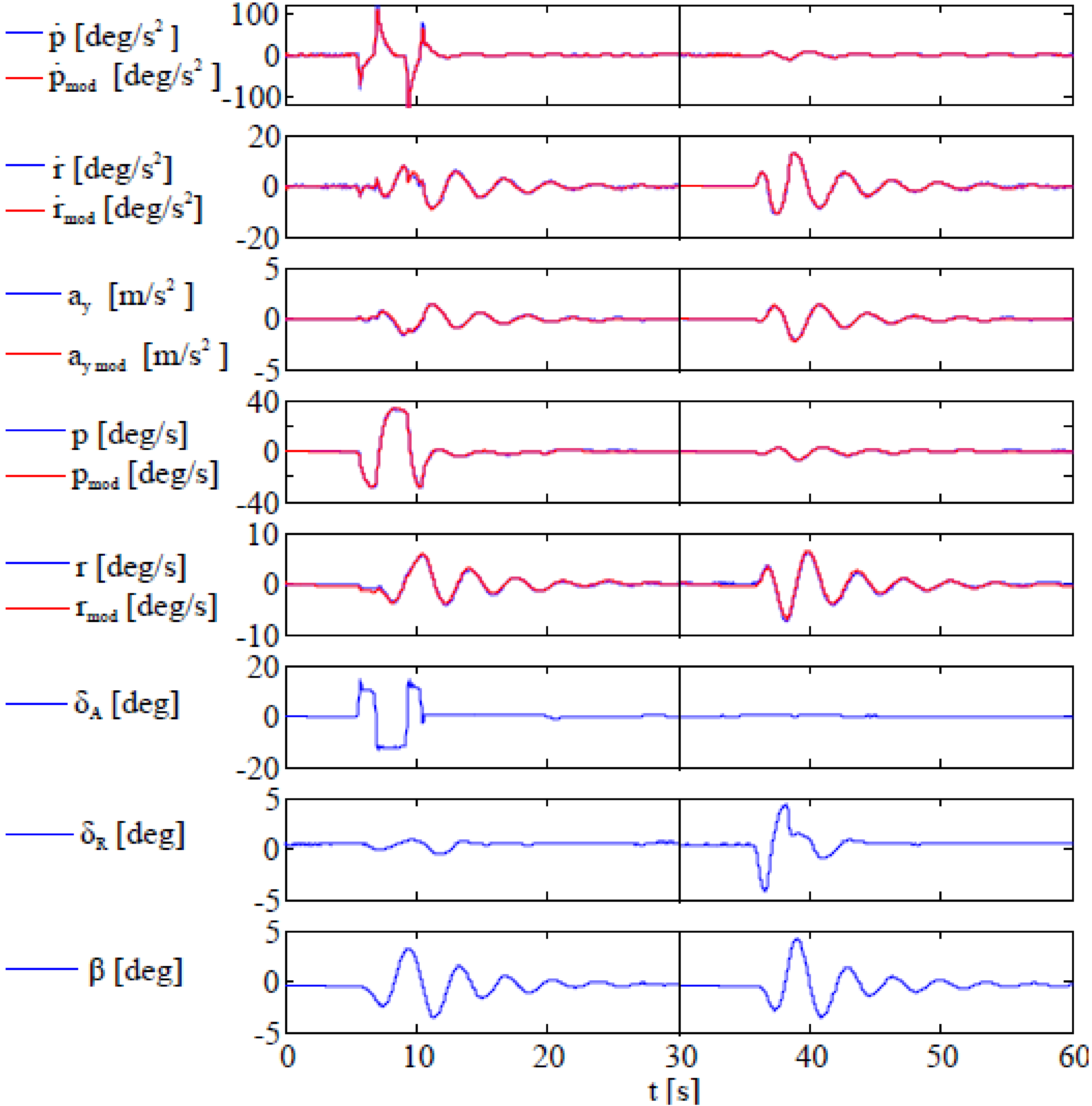
- VFW 614 ATTAS – lateral motion
  - Bias – initial conditions + sensors systematic errors
  - Sideslip angle used as a pseudo-input
  - State equation

$$\begin{bmatrix} \Delta\dot{P} \\ \Delta\dot{R} \end{bmatrix} = \begin{bmatrix} L_P{}' & L_R{}' \\ N_P{}' & N_R{}' \end{bmatrix} \begin{bmatrix} \Delta P \\ \Delta R \end{bmatrix} + \begin{bmatrix} L_{\delta_A}{}' & L_{\delta_R}{}' & L_\beta{}' \\ N_{\delta_A}{}' & N_{\delta_R}{}' & N_\beta{}' \end{bmatrix} \begin{bmatrix} \Delta\delta_A \\ \Delta\delta_R \\ \Delta\beta \end{bmatrix} + \begin{bmatrix} b_{x_{\Delta\dot{P}}} \\ b_{x_{\Delta\dot{R}}} \end{bmatrix}$$

  - Output equation

$$\begin{bmatrix} \Delta\dot{P} \\ \Delta\dot{R} \\ \Delta a_y \\ \Delta P \\ \Delta R \end{bmatrix} = \begin{bmatrix} L'_P & L'_R \\ N'_P & N'_R \\ Y_P & Y_R \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta P \\ \Delta R \end{bmatrix} + \begin{bmatrix} L'_{\delta_A} & L'_{\delta_R} & L'_\beta \\ N'_{\delta_A} & N'_{\delta_R} & N'_\beta \\ Y_{\delta_A} & Y_{\delta_R} & Y_\beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_A \\ \Delta\delta_R \\ \Delta\beta \end{bmatrix} + \begin{bmatrix} b_{y_{\Delta\dot{P}}} \\ b_{y_{\Delta\dot{R}}} \\ b_{y_{\Delta a_y}} \\ b_{y_{\Delta P}} \\ b_{y_{\Delta R}} \end{bmatrix}$$

Example (cont.)



Example (cont.)

Param.	Θ	σ
L_P'	-2,0467	0,0045
L_R'	1,0850	0,0114
L_{δ_A}'	-6,2817	0,0109
L_{δ_R}'	1,1669	0,0287
L_β'	-3,7474	0,0188
N_P'	-0,1735	0,0017
N_R'	-0,4624	0,0045
N_{δ_A}'	-0,3662	0,0042
N_{δ_R}'	-1,7055	0,0111
N_β'	2,9909	0,0079
Y_P	0,7170	0,0195
Y_R	3,9012	0,0541
Y_{δ_A}	1,4464	0,0443
Y_{δ_R}	5,7624	0,1343
Y_β	-27,0588	0,0866

Param.	Time section 1		Time section 2	
	Θ	σ	Θ	σ
$b_{y_{\Delta\dot{P}}}$	-0,0065	0,0038	0,0243	0,0038
$b_{y_{\Delta\dot{R}}}$	0,0293	0,0005	0,0313	0,0005
$b_{y\ \Delta P}$	-0,0065	0,0039	0,0244	0,0039
$b_{y\ \Delta R}$	0,0295	0,0006	0,0317	0,0006
$b_{y\ \Delta a_y}$	-0,1901	0,0043	-0,1764	0,0044

$J=7.04196e-20$

MEL